

MA 221 Homework Solutions

Due date: February 17, 2009

Section 4.5 pg. 192 2b, 5, 7, 21, 27, 29

(Underlined Problems are to be handed in)

2.) Given that $y_1(t) = (1/4) \sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t$, use the superposition principle to find solutions of

(b) $y'' + 2y' + 4y = 2t - 3 \cos 2t$

Since the right hand side of this last equation is twice the right hand side of the second equation given plus -3 times the right hand side of the first equation given a particular solution for (b) is

$$2y_2 - 3y_1 = \frac{t}{2} - \frac{1}{4} - \frac{3}{4} \sin 2t$$

In problems 5 & 7 a nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

5.) $\theta'' - \theta' - 2\theta = 1 - 2t$ $\theta_p(t) = t - 1$

The homogeneous equation is

$$\theta'' - \theta' - 2\theta = 0$$

Thus $r^2 - r - 2 = (r - 2)(r + 1) = 0$ so $r = 2, -1$

$$y_h = c_1 e^{2t} + c_2 e^{-t}$$

Therefore by the superposition principle:

$$y = \theta_h + \theta_p = t - 1 + c_1 e^{2t} + c_2 e^{-t}$$

SNB check: $\theta'' - \theta' - 2\theta = 1 - 2t$, (Compute \rightarrow Solve ODE \rightarrow Exact \rightarrow Independent

Variable t)

Exact solution is: $t + C_3 e^{-t} + C_4 e^{2t} - 1$

7.) $y'' = 2y' - y + 2e^x$ $y_p(x) = x^2 e^x$

The homogeneous equation is

$$y'' - 2y' + y = 0$$

Thus $r^2 - 2r + 1 = (r - 1)^2 = 0$ so $r = 1$ is a repeated root and

$$y_h = c_1 e^x + c_2 x e^x$$

Therefore

$$y = y_h + y_p = c_1 e^x + c_2 x e^x + x^2 e^x$$

SNB check: $y'' - 2y' + y = 2e^x$ (Compute \rightarrow Solve ODE \rightarrow Exact \rightarrow Independent Variable

x)

Exact solution is: $C_1 e^x + C_2 x e^x + x^2 e^x$

In problem 21, find a general solution to the differential equation.

21.) $y''(\theta) + 2y'(\theta) + 2y(\theta) = e^{-\theta} \cos \theta$

Let $p(r) = r^2 + 2r + 2 = 0$

$$r = -1 \pm i$$

$$y_h(\theta) = c_1 e^{-\theta} \cos \theta + c_2 e^{-\theta} \sin \theta = (c_1 \cos \theta + c_2 \sin \theta) e^{-\theta}$$

We now find a particular solution.

Method 1:

$$y_p(\theta) = \theta(A \cos \theta + B \sin \theta)e^{-\theta}$$

$$y_p'(\theta) = (A \cos \theta + B \sin \theta)e^{-\theta} + \theta[(A \cos \theta + B \sin \theta)e^{-\theta}]'$$

$$y_p''(\theta) = 2[(A \cos \theta + B \sin \theta)e^{-\theta}]' + \theta[(A \cos \theta + B \sin \theta)e^{-\theta}]'' = 2[(B - A) \cos \theta - (B + A) \sin \theta]$$

$$y_p'' + 2y_p' + 2y_p = 2[(B - A) \cos \theta - (B + A) \sin \theta]e^{-\theta} + 2(A \cos \theta + B \sin \theta)e^{-\theta} = 2(B \cos \theta - A$$

$$A = 0, B = 1/2, \text{ so } y_p(\theta) = (1/2)\theta e^{-\theta} \sin \theta$$

Method 2: Consider a companion $v''(\theta) + 2v'(\theta) + 2v(\theta) = e^{-\theta} \sin \theta$. Multiply this equation by i and add it to the original equation. Letting $w = y + iv$, we have

$$w'' + 2w' + 2w = e^{-\theta}(\cos \theta + i \sin \theta) = e^{-\theta} e^{i\theta} = e^{(i-1)\theta}$$

Then $p(i-1) = (i-1)^2 + 2(i-1) + 2 = -2i + 2i - 1 + 2 = 0$. $p'(r) = 2r + 2$ so $p(i-1) = 2(i-1) + 2 = 2i$

Therefore

$$w_p = -\frac{\theta e^{(i-1)\theta}}{2} = \frac{\theta e^{-\theta}(\cos \theta + i \sin \theta)}{2i} = -\frac{1}{2}\theta e^{-\theta}(i \cos \theta - \sin \theta)$$

Thus

$$y_p = \operatorname{Re} w_p = \frac{1}{2}\theta e^{-\theta} \sin \theta$$

Therefore

$$y = y_h + y_p = c_1 e^{-\theta} \cos \theta + c_2 e^{-\theta} \sin \theta + \frac{1}{2}\theta e^{-\theta} \sin \theta$$

In problems 27 and 29, find the solution to the Initial Value Problem.

$$27.) y''(x) - y'(x) - 2y(x) = \cos x - \sin 2x; \quad y(0) = -7/20, \quad y'(0) = 1/5$$

$$p(r) = r^2 - r - 2 = 0 = (r-2)(r+1) \Rightarrow r = -1; r = 2$$

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x}$$

Method 1:

To find a particular solution for $\cos x$ we consider the two equations

$$y'' - y' - 2y = \cos x$$

$$v'' - v' - 2v = \sin x$$

Multiply the second equation by i and add it to the first equation and let $w = y + iv$ to get

$$w'' - w' - 2w = \cos \theta + i \sin \theta = e^{i\theta}$$

Since $p(i) = -1 - i - 2 = -3 - i \neq 0$ we have that

$$\begin{aligned} w_p &= \frac{ke^{i\theta}}{p(i)} = \frac{e^{i\theta}}{-3-i} = -\frac{1}{3+i} \frac{(3-i)}{(3-i)} (\cos \theta + i \sin \theta) \\ &= \frac{i-3}{10} (\cos \theta + i \sin \theta) \end{aligned}$$

Therefore $y_{p_1} = \operatorname{real part of } w_p = -(3/10) \cos x - (1/10) \sin x$

To find a particular solution for $-\sin 2x$ we consider the two equations

$$y'' - y' - 2y = -\sin 2x$$

$$v'' - v' - 2v = -\cos 2x$$

Multiply the second equation by i and add it to the first equation and let $w = v + iy$ to get

$$w'' - w' - 2w = -\cos 2x - i \sin 2x = -e^{2ix}$$

Since $p(2i) = -4 - 2i - 2 = -6 - 2i \neq 0$ we have that

$$\begin{aligned} w_p &= \frac{ke^{\alpha\theta}}{p(\alpha)} = \frac{-e^{2ix}}{-6 - 2i} = \frac{1}{6 + 2i} \frac{(6 - 2i)}{(6 - 2i)} (\cos 2x + i \sin 2x) \\ &= \frac{2(3 - i)}{40} (\cos 2x + i \sin 2x) \end{aligned}$$

Therefore y_{p_2} =imaginary part of $w_p = -(1/20) \cos 2x + (3/20) \sin 2x$

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{10} \cos x - \frac{1}{10} \sin x - \frac{1}{20} \cos 2x + \frac{3}{20} \sin 2x$$

SNB check:

$$\begin{aligned} &y'' - y' - 2y = \cos x - \sin 2x, \text{ Exact solution is:} \\ &\left\{ \frac{3}{20} \sin 2x - \frac{1}{10} \sin x - \frac{1}{20} \cos 2x - \frac{3}{10} \cos x + C_2 e^{-x} + C_3 e^{2x} \right\} \end{aligned}$$

Method 2:

For the equation $y'' - y' - 2y = \cos x$

$$y_{p,1}(x) = A \cos x + B \sin x$$

$$\begin{aligned} y'' - y' - 2y &= (-A \cos x - B \sin x) - (-A \sin x + B \cos x) - 2(A \cos x + B \sin x) \\ &= (-3A - B) \cos x + (A - 3B) \sin x = \cos x \end{aligned}$$

$$A = -3/10; B = -1/10$$

So, $y_{p_1}(x) = -(3/10) \cos x - (1/10) \sin x$

For the equation $y'' - y' - 2y = -\sin 2x$

$$y_{p_2}(x) = A \cos 2x + B \sin 2x$$

$$\begin{aligned} y'' - y' - 2y &= (-4A \cos 2x - 4B \sin 2x) - (-2A \sin 2x + 2B \cos 2x) - 2(A \cos 2x + B \sin 2x) \\ &= (-6A - 2B) \cos 2x + (2A - 6B) \sin 2x = -\sin 2x \end{aligned}$$

$$A = -1/20; B = +3/20$$

So, $y_{p_2}(x) = -(1/20) \cos 2x + (3/20) \sin 2x$

so again we have

$$\begin{aligned} y &= y_h + y_{p_1} + y_{p_2} = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{10} \cos x - \frac{1}{10} \sin x - \frac{1}{20} \cos 2x + \frac{3}{20} \sin 2x \\ y' &= -c_1 e^{-x} + 2c_2 e^{2x} + \frac{3}{10} \sin x - \frac{1}{10} \cos x + \frac{1}{10} \sin 2x + \frac{3}{10} \cos 2x \end{aligned}$$

Next, we find c_1 and c_2 such that the initial conditions are satisfied.

$$y(0) = c_1 + c_2 - \frac{3}{10} - \frac{1}{20} = -\frac{7}{20}$$

$$y'(0) = -c_1 + 2c_2 - \frac{1}{10} + \frac{3}{10} = \frac{1}{5}$$

Thus

$$\begin{aligned}c_1 + c_2 &= 0 \\ -c_1 + 2c_2 &= 0\end{aligned}$$

Hence $c_1 = c_2 = 0$

With these constants the solution is,

$$y(x) = -\frac{3}{10} \cos x - \frac{1}{10} \sin x - \frac{1}{20} \cos 2x + \frac{3}{20} \sin 2x$$

29.) $y''(\theta) - y(\theta) = \sin \theta - e^{2\theta}$ $y(0) = 1$, $y'(0) = -1$

The characteristic equation is $p(r) = r^2 - 1 = 0$, so $r = \pm 1$ and $y_h = c_1 e^\theta + c_2 e^{-\theta}$.

Method 1:

To find a particular solution for $\sin \theta$ we consider the two equations

$$v'' - v = \cos \theta$$

$$y'' - y = \sin \theta$$

Multiply the second equation by i and add it to the first equation and let $w = v + iy$ to get

$$w'' - w = \cos \theta + i \sin \theta = e^{i\theta}$$

Since $p(i) = -2 \neq 0$ we have that

$$w_p = \frac{ke^{i\theta}}{p(i)} = \frac{e^{i\theta}}{-2} = -\frac{1}{2}(\cos \theta + i \sin \theta)$$

Therefore y_{p1} =imaginary part of $w_p = -\frac{1}{2} \sin \theta$

To find a particular solution for $-e^{2\theta}$ note that $p(2) = (2)^2 - 1 = 3$ so

$$y_{p2} = \frac{-e^{2\theta}}{3}$$

A general solution is then

$$y = y_h + y_{p1} + y_{p2} = c_1 e^\theta + c_2 e^{-\theta} - \frac{1}{2} \sin \theta - \frac{e^{2\theta}}{3}$$

Method 2:

For the equation $y'' - y = \sin \theta$

$$y_{p,1} = A \cos \theta + B \sin \theta$$

$$(-A \cos \theta - B \sin \theta) - (A \cos \theta + B \sin \theta) = -2A \cos \theta - 2B \sin \theta = \sin \theta$$

$$-2A = 0$$

$$-2B = 1$$

$$A = 0, B = -1/2$$

$$\text{So, } y_{p,1} = -(1/2) \sin \theta$$

For the equation $y'' - y = e^{2\theta}$

$$y_{p,2} = Ae^{2\theta}$$

$$3Ae^{2\theta} = e^{2\theta} \Rightarrow A = 1/3$$

$$y_{p,2} = (1/3)e^{2\theta}$$

$$y = y_h + y_{p1} + y_{p2} = c_1 e^\theta + c_2 e^{-\theta} - \frac{1}{2} \sin \theta - \frac{e^{2\theta}}{3}$$

SNB check: $y'' - y = \sin \theta - e^{2\theta}$, Exact solution is: $C_{10}e^\theta - \frac{1}{2} \sin \theta - \frac{1}{3}e^{2\theta} + C_9e^{-\theta}$

The initial conditions imply since $y' = c_1 e^\theta - c_2 e^{-\theta} - \frac{1}{2} \cos \theta - \frac{2}{3} e^{2\theta}$

$$y(0) = c_1 + c_2 - \frac{1}{3} = 1$$

$$y'(0) = c_1 - c_2 - \frac{1}{2} - \frac{2}{3} = -1$$

$$c_1 + c_2 = \frac{4}{3}$$

$$c_1 - c_2 = \frac{1}{6}$$

Hence $c_1 = \frac{3}{4}$, $c_2 = \frac{7}{12}$ Thus

$$y = \frac{3}{4}e^\theta + \frac{7}{12}e^{-\theta} - \frac{1}{2}\sin\theta - \frac{e^{2\theta}}{3}$$

$$y'' - y = \sin\theta - e^{2\theta}$$

SNB check: $y(0) = 1$, Exact solution is: $\frac{3}{4}e^\theta - \frac{1}{2}\sin\theta + \frac{7}{12}e^{-\theta} - \frac{1}{3}e^{2\theta}$

$$y'(0) = -1$$