

MA 221 Homework Solutions

Due date: February 19-20, 2009

Section 4.6 pg 197-98: 1, 3, 11, 18, 21

Section 8.5 pg 460 1, 3, 5, 15

(Underlined problems are to be handed in)

Section 4.6 pg 197-98

1.) Find a general solution to the differential equation using variation of parameters.

$$y'' + 4y = \tan 2t$$

This equation has associated homogenous equation

$$y'' + 4y = 0$$

The roots of the associated homogenous equation, $r^2 + 4 = 0$, are $r = \pm 2i$. Therefore, a general solution to this equation is

$$y_h(t) = c_1 \cos 2t + c_2 \sin 2t$$

For the variation of parameters method, we let

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \quad \text{where} \quad y_1(t) = \cos 2t \quad \text{and} \quad y_2(t) = \sin 2t$$

Thus, $y_1'(t) = -2 \sin 2t$ and $y_2'(t) = 2 \cos 2t$. This means that we have to solve the system

$$v_1' \cos 2t + v_2' \sin 2t = 0$$

$$-2v_1'(t) \sin 2t + 2v_2'(t) \cos 2t = \tan 2t$$

Multiplying the first equation by $\sin 2t$ the second equation by $\frac{1}{2} \cos 2t$ and adding the resulting equations together, we get:

$$v_2'(t) = \frac{1}{2} \sin 2t$$

$$v_2 = \frac{1}{2} \int \sin 2t dt = -\frac{1}{4} \cos 2t + c_3$$

Substituting v_2' into the first equation ($v_1' \cos 2t + v_2' \sin 2t = 0$), we get

$$v_1'(t) = -v_2'(t) \tan 2t = -\frac{1}{2} \frac{\sin^2 2t}{\cos 2t} = -\frac{1}{2} \frac{1 - \cos^2 2t}{\cos 2t}$$

$$= -\frac{1}{2} (\cos 2t - \sec 2t)$$

$$\Rightarrow v_1(t) = \frac{1}{2} \int (\cos 2t - \sec 2t) dt = \frac{1}{4} (\sin 2t - \ln|\sec 2t + \tan 2t|) + c_4$$

We take $c_3 = c_4 = 0$ since we just need a particular solution. Thus,

$$y_p(t) = \frac{1}{4} (\sin 2t - \ln|\sec 2t + \tan 2t|) \cos 2t - \frac{1}{4} \cos 2t \sin 2t$$

$$= -\frac{1}{4} \cos 2t \ln|\sec 2t + \tan 2t|$$

and general solution is

$$y_p(t) = y_h(t) + y_p(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4} \cos 2t \ln|\sec 2t + \tan 2t|$$

3.) Find a general solution to the differential equation using variation of parameters.

$$2x'' - 2x' - 4x = 2e^{3t}$$

First, we can simplify the equation by dividing both sides by 2. This yields

$$x'' - x' - 2x = e^{3t}$$

This equation has associated homogenous equation

$$x'' - x' - 2x = 0$$

The roots of the associated homogenous equation, $r^2 - r - 2 = 0$, are $r = 2$ and $r = -1$. Therefore, a general solution to this equation is

$$x_h(t) = c_1 e^{2t} + c_2 e^{-t}$$

For the variation of parameters method, we let

$$x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t) \text{ where } x_1(t) = e^{2t} \text{ and } x_2(t) = e^{-t}$$

Thus, $x_1'(t) = 2e^{2t}$ and $x_2'(t) = -e^{-t}$. This means that we have to solve the

system

$$e^{2t}v_1' + e^{-t}v_2' = 0$$

$$2e^{2t}v_1' - e^{-t}v_2' = e^{3t}$$

Adding this two equations yields

$$3e^{2t}v_1' = e^{3t}$$

$$\Rightarrow v_1' = \frac{1}{3}e^t$$

$$\Rightarrow v_1(t) = \frac{1}{3}e^t$$

Substituting v_1' into the first equation, we get

$$\frac{1}{3}e^{3t} + e^{-t}v_2' = 0$$

$$\Rightarrow v_2' = -\frac{1}{3}e^{4t}$$

$$\Rightarrow v_2(t) = -\frac{1}{12}e^{4t}$$

Therefore,

$$x_p(t) = \frac{1}{3}e^t e^{2t} - \frac{1}{12}e^{4t} e^{-t} = \frac{1}{4}e^{3t}$$

and general solution is

$$x(t) = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{4}e^{3t}.$$

11.) Find a general solution to the differential equation

$$y'' + y = \tan^2 t$$

This equation is similar to that in Example 1 on page 196 in the text. Only the homogeneous term is different. Thus we will follow steps in Example 1. Two independent solutions to the corresponding homogenous equation, $y'' + y = 0$, are

$y_1(t) = \cos t$ and $y_2(t) = \sin t$. A particular solution to the original equation is of

the form

$$y_p(t) = v_1(t) \cos t + v_2(t) \sin t,$$

where $v_1(t)$ and $v_2(t)$ satisfy

$$v_1'(t) \cos t + v_2'(t) \sin t = 0$$

$$-v_1'(t) \sin t + v_2'(t) \cos t = \tan^2 t,$$

Multiplying the first equation by $\sin t$ and the second equation by $\cos t$, and adding them together yield

$$v_2'(t) = \tan^2 t \cos t = (\sec^2 t - 1) \cos t = \sec t - \cos t$$

We find $v_1'(t)$ from the first equation in the system.

$$v_1'(t) = -v_2'(t) \tan t = -(\sec t - \cos t) \tan t = \sin t - \frac{\sin t}{\cos^2 t}$$

Integrating, we get

$$v_1(t) = \int (\sin t - \frac{\sin t}{\cos^2 t}) dt = -\cos t - \sec t$$

$$v_2(t) = \int (\sec t - \cos t) dt = \ln|\sec t + \tan t| - \sin t$$

where we take zero integrating constants. Therefore,

$$y_p(t) = -(\cos t + \sec t) \cos t + (\ln|\sec t + \tan t| - \sin t) \sin t = \sin t \ln|\sec t + \tan t| - 2,$$

and a general solution is given by

$$y(t) = c_1 \cos t + c_2 \sin t + \sin t \ln|\sec t + \tan t| - 2.$$

18.) Find a general solution to the DE $y'' - 6y' + 9y = t^{-3}e^{3t}$

The characteristic equation is $p(r) = r^2 - 6r + 9 = (r - 3)^2 = 0$ so $r = 3$ is a repeated root and

$$y_h = c_1 e^{3t} + c_2 t e^{-3t}$$

Hence $y_p = v_1 e^{3t} + v_2 t e^{3t}$, and we have

$$v_1' e^{3t} + v_2' t e^{3t} = 0$$

$$3v_1' e^{3t} + v_2' (3t e^{3t} + e^{3t}) = \frac{e^{3t}}{t^3}$$

The equations for v_1' and v_2' are

$$v_1' = \frac{\begin{vmatrix} 0 & t e^{3t} \\ \frac{e^{3t}}{t^3} & 3t e^{3t} + e^{3t} \end{vmatrix}}{\begin{vmatrix} e^{3t} & t e^{3t} \\ 3e^{3t} & 3t e^{3t} + e^{3t} \end{vmatrix}} = \frac{-\frac{e^{6t}}{t^2}}{e^{6t}} = -\frac{1}{t^2}$$

$$v_2' = \frac{\begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & \frac{e^{3t}}{t^3} \end{vmatrix}}{\begin{vmatrix} e^{3t} & t e^{3t} \\ 3e^{3t} & 3t e^{3t} + e^{3t} \end{vmatrix}} = \frac{\frac{e^{6t}}{t^3}}{e^{6t}} = \frac{1}{t^3}$$

$$v_1 = t^{-1}$$

$$v_2 = -\frac{1}{2} t^{-2}$$

$$y_p = v_1 e^{3t} + v_2 t e^{3t}$$

$$= t^{-1} e^{3t} - \frac{1}{2} t^{-1} e^{3t} = \frac{1}{2} t^{-1} e^{3t}$$

Hence

$$y = y_h + y_p = c_1 e^{3t} + c_2 t e^{-3t} + \frac{1}{2} t^{-1} e^{3t}$$

SNB check: $y'' - 6y' + 9y = t^{-3} e^{3t}$, Exact solution is: $y(t) = \frac{1}{2t} e^{3t} + C_1 e^{3t} + C_2 e^{3t} t$

21.) Find a particular solution to the variable coefficient equation

$$t y'' - (t + 1) y' + y = t^2$$

given that

$$y_1 = e^t \quad y_2 = t + 1$$

are linearly independent solutions to the corresponding homogeneous equation for $t > 0$.

We let $y_p = v_1 e^t + v_2 (t + 1)$. Then the equations for v_1' and v_2' are

$$v_1' e^t + v_2' (t + 1) = 0$$

$$v_1' e^t + v_2' = \frac{f}{a} = \frac{t^2}{t} = t$$

Then

$$v_1' = \frac{\begin{vmatrix} 0 & t+1 \\ t & 1 \end{vmatrix}}{\begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix}} = \frac{-t(t+1)}{-te^t} = (t+1)e^{-t}$$

$$v_2' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & t \end{vmatrix}}{\begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix}} = \frac{te^t}{-te^t} = -1$$

Hence using SNB to evaluate the integral for v_1

$$v_1 = \int (t+1)e^{-t} dt = -(t+2)e^{-t}$$

$$v_2 = -t$$

and

$$y_p = v_1 e^t + v_2(t+1) = -(t+2)e^{-t}e^t - t(t+1) = -t^2 - 2t - 2$$

Since $-2t - 2 = -2(t+1)$ this term is a homogeneous solution so

$$y_p = -t^2$$

Section 8.5 pg 460

1.) Use the substitution $y = x^r$ to find a general solution to the equation

$$x^2 y''(x) + 6xy'(x) + 6y(x) = 0$$

$$y(x) = x^r$$

$$y'(x) = rx^{r-1}$$

$$y''(x) = r(r-1)x^{r-2}$$

$$x^2 r(r-1)x^{r-2} + 6rx^{r-1} + 6y = 0$$

$$(r^2 + 5r + 6) = 0$$

$$(r+3)(r+2) = 0$$

$$r = -2, r = -3$$

$$y(x) = c_1 x^{-2} + c_2 x^{-3}$$

3.) Use the substitution $y = x^r$ to find a general solution to the equation

$$x^2 y''(x) - xy'(x) + 17y(x) = 0$$

$$y(x) = x^r$$

$$y'(x) = rx^{r-1}$$

$$y''(x) = r(r-1)x^{r-2}$$

$$x^2 [r(r-1)x^{r-2}] - x(rx^{r-1}) + 17x^r = 0$$

$$x^r (r^2 - 2r + 17) = 0$$

$$(r^2 - 2r + 17) = 0$$

$$r = \frac{2 - \sqrt{2^2 - 4(1)(17)}}{2} = 1 \pm 4i$$

$$y_1(x) = x \cos(4 \ln x), y_2(x) = x \sin(4 \ln x)$$

$$y(x) = c_1 x \cos(4 \ln x) + c_2 x \sin(4 \ln x)$$

5.) Use the substitution $y = x^r$ to find a general solution to $\frac{d^2 y}{dx^2} = \frac{5}{x} \frac{dy}{dx} - \frac{13}{x^2} y$

Notice that, since $x > 0$, we can multiply this differential equation by x^2 and rewrite it to obtain

$$x^2 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} + 13y = 0$$

$$y(x) = x^r$$

$$y'(x) = rx^{r-1}$$

$$y''(x) = r(r-1)x^{r-2}$$

$$r(r-1)x^r - 5rx^r + 13x^r = 0$$

$$(r^2 - 6r + 13)x^r = 0$$

$$(r^2 - 6r + 13) = 0$$

$$r = \frac{6 + \sqrt{36 - 52}}{2} = 3 - 2i$$

$$y_1(x) = x^3 \cos(2 \ln x), y_2(x) = x^3 \sin(2 \ln x)$$

$$y(x) = c_1 x^3 \cos(2 \ln x) + c_2 x^3 \sin(2 \ln x)$$

15.) Solve the initial value problem

$$t^2 x(t) - 12x(t) = 0 \quad x(1) = 3, \quad x'(1) = 5$$

The indicial equation is

$$r^2 + (p-1)r + q = r^2 - r - 12 = (r-4)(r+3) = 0$$

Thus $r = 4, -3$ and

$$x(t) = c_1 t^4 + c_2 t^{-3}$$

The initial conditions imply since $x'(t) = 4c_1 t^3 - 3c_2 t^{-4}$

$$x(1) = c_1 + c_2 = 3$$

$$x'(1) = 4c_1 - 3c_2 = 5$$

Therefore $c_1 = 2, c_2 = 1$ so that

$$x(t) = 2t^4 + t^{-3}$$