

MA 221 Homework Solutions

Due date: February 26/27, 2009

7.3 p. 365 #1, 3, 11, 21, 25a,b, 31, 33;
 (Underlined problems are to be turned in)

For problems 1, 3 and 11, determine the Laplace transform of the given function using Table 7.1 and the properties of the transform given in Table 7.2.

1.) $t^2 + e^t \sin 2t$

By the linearity of the Laplace transform,

$$\mathcal{L}\{t^2 + e^t \sin 2t\}(s) = \mathcal{L}\{t^2\}(s) + \mathcal{L}\{e^t \sin 2t\}(s)$$

$$\mathcal{L}\{t^2\}(s) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}, s > 0;$$

$$\mathcal{L}\{e^t \sin 2t\}(s) = \frac{2}{[s-1]^2 + 2^2}, s > 1;$$

$$\mathcal{L}\{t^2 + e^t \sin 2t\}(s) = \frac{2}{s^3} + \frac{2}{(s-1)^2 + 4}$$

3.) $e^{-t} \cos 3t + e^{6t} - 1$

By the linearity of the Laplace transform,

$$\mathcal{L}\{e^{-t} \cos 3t + e^{6t} - 1\}(s) = \mathcal{L}\{e^{-t} \cos 3t\}(s) + \mathcal{L}\{e^{6t}\}(s) - \mathcal{L}\{1\}$$

$$\mathcal{L}\{e^{-t} \cos 3t\}(s) = \frac{s-(-1)}{[s-(-1)]^2 + 3^2} = \frac{s+1}{(s+1)^2 + 9}, s > 1;$$

$$\mathcal{L}\{e^{6t}\}(s) = \frac{1}{s-6}, s > 6;$$

$$\mathcal{L}\{1\} = \frac{1}{s}, s > 0.$$

$$\mathcal{L}\{e^{-t} \cos 3t + e^{6t} - 1\}(s) = \frac{s+1}{(s+1)^2 + 9} + \frac{1}{s-6} - \frac{1}{s}$$

11.) $\cosh bt$

$$\mathcal{L}\{\cosh bt\}(s) = \mathcal{L}\left\{\frac{e^{bt} + e^{-bt}}{2}\right\}(s) = \frac{1}{2}[\mathcal{L}\{e^{bt}\}(s) + \mathcal{L}\{e^{-bt}\}(s)] = \frac{1}{2}\left[\frac{1}{s-b} + \frac{1}{s+b}\right] = \frac{s}{s^2 - b^2}$$

21.) Given that $\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2 + b^2}$, use the translation property to compute

$$\mathcal{L}\{e^{at} \cos bt\}(s)$$

$$\mathcal{L}\{e^{at} f(t)\}(s) = F(s - a)$$

In this case $f(t) = \cos bt$

$$\text{For } \mathcal{L}\{\cos bt\}(s) \Rightarrow F(s) = \frac{s}{s^2 + b^2} \quad \Rightarrow \quad F(s - a) = \frac{s - a}{(s - a)^2 + b^2}$$

$$\boxed{\mathcal{L}\{e^{at} \cos bt\} = \frac{s - a}{(s - a)^2 + b^2}}$$

25.) Use formula (6) to help determine

(a) $\mathcal{L}[t \cos bt]$

By property (6) on pg. 363

$$\mathcal{L}[t \cos bt] = -[\mathcal{L}\{\cos bt\}]' = -\left[\frac{s}{s^2 + b^2}\right]' = \frac{s^2 - b^2}{(s^2 + b^2)^2}$$

$$\boxed{\mathcal{L}[t \cos bt] = \frac{s^2 - b^2}{(s^2 + b^2)^2}}$$

(b) $\mathcal{L}[t^2 \cos bt]$

By property (6) on pg. 363

$$\begin{aligned}\mathcal{L}[t^2 \cos bt] &= \mathcal{L}[t(t \cos bt)] = -\mathcal{L}[t \cos bt]' \\ &= -\left[\frac{s^2 - b^2}{(s^2 + b^2)^2} \right]' = \frac{2s^3 - 6sb^2}{(s^2 + b^2)^3} \\ \mathcal{L}[t^2 \cos bt] &= \frac{2s^3 - 6sb^2}{(s^2 + b^2)^3}\end{aligned}$$

31.) Show that for $c < 0$, the translated function

$$g(t) = \begin{cases} 0, & 0 < t < c \\ f(t - c), & c < t \end{cases}$$

has the Laplace transform

$$\begin{aligned}\mathcal{L}\{g\}(s) &= e^{-cs} \mathcal{L}\{f\}(s) \\ \mathcal{L}\{g(t)\}(s) &= \int_0^\infty e^{-st} g(t) dt = \int_0^c (0) dt + \int_c^\infty e^{-st} f(t - c) dt = \quad, \text{let } (t - c) = u, dt = du \\ &= \int_0^\infty e^{-s(u+c)} f(u) du = e^{-sc} \int_0^\infty e^{-su} f(u) du = e^{-cs} \mathcal{L}\{f(t)\}(s).\end{aligned}$$

33.) Let $g(t)$ be the given function $f(t)$ translated to the right by c units. Sketch $f(t)$ and $g(t)$ and find $\mathcal{L}\{g(t)\}(s)$

$$\begin{aligned}f(t) &= t, \quad c = 1 \\ \mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t) dt = \int_0^c (0) dt + \int_0^\infty e^{-st} f(t - c) dt = (t - c \rightarrow u, dt \rightarrow du) \\ &= \int_0^\infty e^{-s(u+c)} f(u) du = e^{-cs} \int_0^\infty e^{-su} f(u) du = e^{-cs} \mathcal{L}\{f(t)\} \\ \mathcal{L}\{g(t)\} &= e^{-cs} \mathcal{L}\{f(t)\} \\ \text{with } c &= 1\end{aligned}$$

$$\mathcal{L}\{g(t)\} = e^{-(1)s} \mathcal{L}\{t\} = \frac{e^{-s}}{s^2}$$

Graph:

red line: $g(t)$

black line: $f(t)$

