MA 221 Homework Solutions
Due date: March 24, 2009

8.3 p. 449 #1, 3, 5, 7, 11, 12, 15, 17, 19, 21,25 27
(Underlined problems are to be handed in)

In problems 1, 3, 5 and 7 Determine all the singular points of the given differential equations.

1.) $(x+1)y'' - x^2y' + 3y = 0$
Dividing the entire equation by $(x+1)$ yields

$$y'' - \frac{x^2}{x+1}y' + \frac{3}{x+1}y = 0$$

We then see:

$$P(y) = -\frac{x^2}{x+1} \qquad Q(y) = \frac{3}{x+1}$$

These are rational functions and so they are analytical everywhere except, perhaps, at zeros of their denominators. Solving $x+1=0$ we find that $x=-1$ which is at a point of infinite discontinuity for both functions. Consequently, $x=-1$ is the only singular point of the given equation.

3.) $(\theta^2-2)y'' + 2y' + \sin\theta y = 0$
Writing the equation in standard form yields

$$y'' + \frac{2}{\theta^2-2}y' + \frac{\sin\theta}{\theta^2-2}y = 0$$

and

$$P(\theta) = \frac{2}{\theta^2-2} \qquad Q(\theta) = \frac{\sin\theta}{\theta^2-2}$$

The singularities are therefore at $\theta = \pm\sqrt{2}$.

Find at least the first four nonzero terms in a power series expansion about $x=0$ for a general solution to the given differential equation.

5.) $(t^2-t-2)x'' + (t+1)x' - (t-2)x = 0$

$$x'' + \frac{t+1}{t^2-t-2}x' - \frac{t-2}{t^2-t-2}x = 0$$

$$p(t) = \frac{t+1}{t^2-t-2} = \frac{t+1}{(t+1)(t-2)}$$

$$q(t) = \frac{t-2}{t^2-t-2} = \frac{t-2}{(t+1)(t-2)}$$

The point $t=-1$ is a removable singularity for $p(t)$ since, for $t \neq -1$, we can cancel $(t+1)$ term in the numerator and denominator, and so $p(t)$ becomes analytic at $t=-1$ if we set

$p(-1) := \lim_{t\to -1} p(t) = \lim_{t\to -1} \frac{1}{t-2} = -\frac{1}{3}$

At the point $t = 2, p(t)$ has infinite discontinuity. Thus $p(t)$ is analytic everywhere except $t = 2.$ Similarly, $q(t)$ is analytic everywhere except $t = -1.$ Therefore, the given equation has two singular points, $t = -1$ and $t = 2.$

7.) $(\sin x)y'' + (\cos x)y = 0$

Putting the equation in standard form we get:

$y'' + \frac{(\cos x)}{(\sin x)}y = 0$ Hence:

$p(x) = 0 \quad q(x) = \frac{(\cos x)}{(\sin x)} = \cot x$

Since the cotangent function is ±∞ at integer multiples of π, we see that $q(x)$ is not defined and , therefore not analytical at $n\pi$. Hence the differential equation is singular only at the points πn, where n is an integer.

In problems 11, 12, 15 and 17 find at least the first four non zero terms in a power series expansion about $x = 0$ for a general solution to the given differential equation.

11.) $y' + (x+2)y = 0$

The coefficient, $x+2$, is a polynomial, and so it is analytical everywhere. Therefore, $x = 0$ is an ordinary point on the given equation.

We seek the power series solutiin in the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \Rightarrow \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

We now substitute the power series for y and y' into the given differential equation and obtain:

$$y' + (x+2)y = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + (x+2)\sum_{n=0}^{\infty} a_n x^n = 0$$

We want to be able to write the left-hand side of this equation as a single power seriess. This will allow us to find expressions for the coefficient of each power of x. Therefore, we first need to shift the indices in each power series above so that they sum over the same powers of x. Thus, we let $k = n-1$ in the first summation and note that this means that $n = k+1$ and that $k = 0$ when $n = 1$. This yield

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} 2a_k x^k + \sum_{k=1}^{\infty} a_{k-1}x^k = 0$$

$$\Rightarrow \left[a_1 + \sum_{k=1}^{\infty}(k+1)a_{k+1}x^k \right] + \left[2a_0 + \sum_{k=1}^{\infty} 2a_k x^k \right] + \sum_{k=1}^{\infty} a_{k-1}x^k = 0$$

$$\Rightarrow (a_1 + 2a_0) + \sum_{k=1}^{\infty}[(k+1)a_{k+1} + 2a_k + a_{k-1}]x^k = 0$$

For the power series on the left hand side to be identically equato to zero, we must have all zero coefficients. Hence,

$(a_1 + 2a_0) = 0$

$(k+1)a_{k+1} + 2a_k + a_{k-1} = 0$ for all $k \geq 1$

This yields:

$a_1 + 2a_0 = 0 \quad \Rightarrow \quad a_1 = -2a_0$

$k = 1 : 2a_2 + 2a_1 + a_0 = 0 \Rightarrow \quad a_2 = \dfrac{(-2a_1 - a_0)}{2} = \dfrac{(4a_0 - a_0)}{2} = \dfrac{3a_0}{2}$

$k = 2 : 3a_3 + 2a_2 + a_0 = 0 \Rightarrow \quad a_3 = \dfrac{(-2a_2 - a_1)}{3} = \dfrac{(-3a_0 + 2a_0)}{3} = \dfrac{-a_0}{3}$

Therefore,

$$y(x) = a_0 - 2a_0x + \frac{3a_0}{2}x^2 - \frac{a_0}{3}x^3 + \ldots\ldots = a_0\left(1 - 2x + \frac{3x^2}{2} - \frac{x^3}{3}\right)$$

12.) $y' - y = 0$

The coefficient of y is the integer -1, which is analytic everywhere. Thus we expect to find a power series solution of the form

$y(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$

Our task is to determine the coefficients a_n.

For this purpose we need the expansion for $y'(x)$ that is given by termwise differentiation of the above equation:

$y'(x) = 0 + a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}$.

We now substitute the series expansion for y and y' and obtain:

$\sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n = 0.$

We want to be able to write the left-hand side of this equation as a single power series. This will allow us to find expressions for the coefficient of each power of x. Therefore, we first need to shift the indices in each power series above so that they sum over the same powers of x. Thus, we let $k = n - 1$ in the first summation and note that this means that $n = k + 1$ and that $k = 0$ when $n = 1$. This yields

$\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k$

In the second power series we need only to replace n with k. Substituting all of these expressions into their appropriate places yields

$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - \sum_{k=0}^{\infty} a_kx^k = 0.$

In order for this power series to equal to zero, each coefficient must be zero. Therefore, we obtain

$(k+1)a_{k+1} - a_k = 0 \quad \rightarrow \quad a_{k+1} = \frac{a_k}{(k+1)}$

Setting $k = 1, 2, 3\ldots$ and using the fact that $a_1 = a_0$

$a_2 = \frac{a_1}{(1+1)} = \frac{a_0}{2} \qquad a_4 = \frac{a_3}{(3+1)} = \frac{a_3}{4} = \frac{1}{4}(\frac{1}{3}(\frac{a_0}{2}))$

$a_3 = \frac{a_2}{(2+1)} = \frac{1}{3}(\frac{a_0}{2}) \qquad a_5 = \frac{a_4}{(4+1)} = \frac{a_4}{5} = \frac{1}{5}(\frac{1}{4}(\frac{1}{3}(\frac{a_0}{2}))) \qquad etc.$

Hence the power series for the solution takes the form

$y(x) = a_0(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots) \qquad = a_0\sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0e^x$

15.)

$$y'' + (x-1)y' + y = 0$$

Here $P = x - 1$ and $Q = 1$ so there are no singularities and $x = 0$ is an ordinary point. Then

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n(n)x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

and the DE implies

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n(n)x^n - \sum_{n=1}^{\infty} a_n(n)x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the first and third sums above by letting $k = n-2$ or $n = k+2$ and $j = n-1$ or $n = j+1$ and get

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{j=0}^{\infty} a_{j+1}(j+1)x^j + \sum_{n=1}^{\infty} a_n(n+1)x^n + a_0 = 0$$

Replacing all of the place keepers by m and writing out the first terms of the first and second sums leads to

$$2(1)a_2 - a_1 + a_0 + \sum_{m=1}^{\infty}[a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) + a_m(m+1)]x^m = 0$$

Thus

$$2a_2 - a_1 + a_0 = 0$$

$$a_{m+2}(m+2)(m+1) - a_{m+1}(m+1) + a_m(m+1) = 0$$

or

$$a_{m+2} = \frac{a_{m+1} - a_m}{m+2} \quad m = 1, 2, 3, \ldots$$

Hence

$$a_2 = \frac{a_1 - a_0}{2}$$

$$m = 1 \Rightarrow a_3 = \frac{a_2 - a_1}{3} = \frac{\frac{a_1 - a_0}{2} - a_1}{3} = \frac{-(a_1 + a_0)}{6}$$

$$m = 2 \Rightarrow a_4 = \frac{a_3 - a_2}{4} = \frac{\frac{-(a_1+a_0)}{6} - \frac{a_1 - a_0}{2}}{4} = \frac{-2a_1 + a_0}{12}$$

Therefore

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \\ &= a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \cdots\right) + a_1\left(x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{6} + \cdots\right) \end{aligned}$$

17.) Find at least the first four nonezero terms ina power series axpansion about $x = 0$ for a general solution to a given differntial equation.

$$w'' - x^2 w' + w = 0$$

$$w(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots = \sum_{n=0}^{\infty} a_nx^n$$

$$w = \sum_{n=0}^{\infty} a_nx^n$$

$$w' = \sum_{n=1}^{\infty} a_n(n)x^{n-1}$$

$$w'' = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

$$w'' - x^2w' + w = \textstyle\sum_{n=0}^{\infty} a_nx^n - x^2\sum_{n=1}^{\infty} a_n(n)x^{n-1} + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

$$\textstyle\sum_{n=0}^{\infty} a_nx^n - \sum_{n=1}^{\infty} a_n(n)x^{n+1} + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} = 0$$

$$k = n-2; n = k+2; n = 2, k = 0; n-1 = k+1$$

$$\textstyle\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

$$k = n+1; n = k-1; n = 1, k = 2$$

$$\textstyle\sum_{n=1}^{\infty} a_n(n)x^{n+1} = \sum_{k=2}^{\infty} a_{k-1}(k-1)x^k$$

$$\textstyle\sum_{n=0}^{\infty} a_nx^n = \sum_{k=0}^{\infty} a_kx^k$$

$$\textstyle\sum_{n=0}^{\infty} a_nx^n - \sum_{n=1}^{\infty} a_n(n)x^{n+1} + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{k=2}^{\infty}$$

$$\textstyle(2)(1)a_2x^0 + (3)(2)a_3x + \sum_{k=2}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{k=2}^{\infty} a_{k-1}(k-1)x^k + a_0x^0 + a_1x + \sum$$

$$\textstyle a_0 + 2a_2 + (6a_3 + a_1)x + \sum_{k=2}^{\infty}[a_{k+2}(k+2)(k+1) - (k-1)a_{k-1} + a_k]x^k = 0$$

$$a_0 = a_0 \qquad a_1 = a_1 \qquad (k+2)(k+1)a_{k+2} - (k-1)a_{k-1} + a_k = 0$$

$$a_0 + 2a_2 = 0 \qquad 6a_3 + a_1 = 0 \qquad a_{k+2} = \frac{(k-1)a_{k-1} - a_k}{(k+2)(k+1)}$$

$$a_2 = -\frac{a_0}{2} \qquad a_3 = -\frac{a_1}{6}$$

$$a_0 = a_0$$

$$a_1 = a_1$$

$$k = 0 : a_2 = -\frac{a_0}{2}$$

$$k = 1 : a_3 = -\frac{a_1}{6}$$

$$k = 2 : a_4 = \frac{(2-1)a_1 - a_2}{(2+1)(2+2)} = \frac{a_1 - a_2}{(4)(3)} = \frac{2a_1 + a_0}{24}$$

$$w(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 =$$

$$= a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{2a_1 + a_0}{24}x^4 =$$

$$= a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{a_1}{12}x^4 + \frac{a_0}{24}x^4 =$$

$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^4}{12}\right)$$

19.) Find a power-series expansion about $x = 0$ for a general solution of

$$y' - 2xy = 0$$

Your answer should include a general formula for the coefficients.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n(n)x^{n-1}$$

so the DE leads to

$$\sum_{n=1}^{\infty} a_n(n)x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Shifting the first sum by letting $k + 1 = n - 1$ or $n = k + 2$ we get

$$\sum_{k=-1}^{\infty} a_{k+2}(k+2)x^{k+1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

or after replacing the place keepers by m

$$a_1 + \sum_{m=0}^{\infty}[a_{m+2}(m+2) - 2a_m]x^{m+1} = 0$$

Thus

$$a_1 = 0$$

$$a_{m+2} = \frac{2}{m+2}a_m \quad m = 0, 1, 2, \ldots$$

Substituting values for m we get

$$m = 0 \Rightarrow a_2 = \frac{2}{2}a_0$$

$$m = 1 \Rightarrow a_3 = \frac{2}{3}a_1 = 0$$

$$m = 2 \Rightarrow a_4 = \frac{2}{4}a_2 = \frac{2^2 a_0}{2 \bullet 4}$$

$$m = 3 \Rightarrow a_5 = 0$$

$$m = 4 \Rightarrow a_6 = \frac{2}{6}a_4 = \frac{2^3 a_0}{2 \bullet 4 \bullet 6}$$

The pattern is therefore

$$a_{2p} = \frac{2^p}{2 \bullet 4 \bullet 6 \cdots (2p)}a_0 = \frac{2^p}{2^p(1 \bullet 2 \bullet 3 \cdots p)}a_0 = \frac{a_0}{p!}$$

and

$$y(x) = a_0 \sum_{p=0}^{\infty} \frac{x^{2p}}{p!}$$

21.) $y'' - xy' + 4y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n(n)x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

Plugging into the DE, we have,

$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - x\sum_{n=1}^{\infty} a_n(n)x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=1}^{\infty} a_n(n)x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$

Now, shift the index of the first sum by letting k=n-2, and we let k=n in the other two power series.

$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} a_k(k)x^k + \sum_{k=0}^{\infty} 4a_k x^k = 0$

Now,

$(2)(1)a_2x^0 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} a_k(k)x^k + 4a_0x^0 + \sum_{k=1}^{\infty} 4a_k x^k = 0$

$\Rightarrow 2a_2 + 4a_0 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} a_k(k)x^k + \sum_{k=1}^{\infty} 4a_k x^k = 0$

$\Rightarrow 2a_2 + 4a_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)a_{k+2} + (-k+4)a_k]x^k = 0$

Now, setting each coefficient of the power series to zero, we see that,

$2a_2 + 4a_0 \Rightarrow a_2 = \frac{-4a_0}{2} = -2a_0,$

$(k+2)(k+1)a_{k+2} + (-k+4)a_k = 0 \Rightarrow a_{k+2} = \frac{(k-4)a_k}{(k+2)(k+1)}, k \geq 1,$

Thus we have:

$k = 1 \Rightarrow a_3 = \frac{-3a_1}{(2)(3)} = -\frac{a_1}{2}$

$k = 2 \Rightarrow a_4 = \frac{-2a_2}{(4)(3)} = \frac{(-2)(-4)a_0}{(4)(3)(2)} = \frac{a_0}{3}$

$k = 3 \Rightarrow a_5 = \frac{-a_3}{(5)(4)} = \frac{(-3)(-1)a_1}{(5)(4)(3)(2)} = \frac{a_1}{40}$

$k = 4 \Rightarrow a_6 = 0$

$k = 5 \Rightarrow a_7 = \frac{a_5}{(7)(6)} = \frac{(-3)(-1)(1)(3)a_1}{(7)(6)(5)(4)(3)(2)} = \frac{a_1}{560}$

$k = 6 \Rightarrow a_8 = \frac{2a_6}{(8)(7)} = 0$

$\Rightarrow a_{n+1} = \frac{(-3)(-1)(1)\ldots(2n-5)a_1}{(2n+1)!}$

Substituting these expressions for the coefficients into the solution

$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$y(x) = a_0 + a_1x - 2a_0x^2 - \frac{a_1}{2}x^3 + \frac{a_0}{3}x^4 + \frac{a_1}{40}x^5 + \ldots + \frac{(-3)(-1)(1)\ldots(2n-5)a_1}{(2n+1)!}x^{2n+1} + \ldots$

$= a_0[1 - 2x^2 + \frac{x^4}{3}] + a_1[x - \frac{x^3}{2} + \frac{x^5}{40} + \frac{(-3)(-1)(1)\ldots(2n-5)a_1}{(2n+1)!}x^{2n+1} + \ldots]$

$= a_0[1 - 2x^2 + \frac{x^4}{3}] + a_1[x + \sum_{k=1}^{\infty} \frac{(-3)(-1)(1)\ldots(2n-5)a_1}{(2n+1)!}x^{2n+1}]$

In problems 25 and 27 find at least the first four non zero terms in a power series expansion about $x = 0$ for a general solution to the given the

Initial Value Problem.

25.) $w'' + 3xw' - w = 0; \qquad w(0) = 2, \qquad w'(0) = 0$

$$w = \sum_{n=0}^{\infty} a_n x^n$$

$$w' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$w'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$w(0) = a_0 = 2, \qquad w'(0) = a_1 = 0$

Plugging into the DE we have

$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + 3x \sum_{n=1}^{\infty} a_n (n) x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} 3a_n (n) x^n - \sum_{n=0}^{\infty} a_n x^n = 0$

Now, shift the index of the first sum by letting k=n-2, and shift the index of the second sum to k=n and we let k=n in the last power series.

$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} 3a_k (k) x^k - \sum_{k=0}^{\infty} a_k x^k = 0$

We can start all of the summations at the same point if we remove the first term in the first and last power series from above. We then have:

$(2a_2 - a_0) + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} + 3k a_k - a_k] x^k = 0$

By equating the coefficients we see all of the terms in the power series must be equal to zero. Then:

$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$

$(k+2)(k+1) a_{k+2} + a_k (3k-1) = 0$

$a_{k+2} = \frac{-a_k (3k-1)}{(k+2)(k+1)} \Rightarrow k \geq 1$

Thus we have,

$k = 1 \Rightarrow a_3 = \frac{-a_1(3-1)}{(1+2)(1+1)} = -\frac{2a_1}{6} = -\frac{a_1}{3}$

$k = 2 \Rightarrow a_4 = \frac{-a_2(3(2)-1)}{(2+2)(2+1)} = -\frac{5a_2}{12} = -\frac{5\left(\frac{a_0}{2}\right)}{12} = -\frac{5a_0}{24}$

$k = 3 \Rightarrow a_5 = \frac{-a_3(3(3)-1)}{(3+2)(3+1)} = \frac{8a_3}{20} = \frac{-8\left(-\frac{a_1}{3}\right)}{20} = \frac{2a_1}{15}$

Substituting these expressions for the coefficients into the solution

$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

yields

$y(x) = a_0 + a_1 x + \frac{a_0}{2} x^2 - \frac{a_1}{3} x^3 - \frac{5a_0}{24} x^4 + \frac{2a_1}{15} x^5 \ldots$

$= a_0[1 + \frac{x^2}{2} - \frac{5}{24}x^4 + \ldots] \ + a_1[x - \frac{1}{3}x^3 + \frac{2}{15}x^5 \ldots]$
$w(0) = a_0 = 2$
$y'(0) = a_1 = 0$
$y(x) = 2 + x^2 - \frac{5}{12}x^4 + \frac{11}{72}x^6 \ldots$

27.) $(x+1)y'' - y = 0; \qquad y(0) = 0, \qquad y'(0) = 1$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

Plugging into the DE we have
$(x+1)\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$
$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$

Now, shift the index of the first sum by letting k=n-2, and shift the index of the second sum to k=n-1 and we let k=n in the last power series.
$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} a_{k+1}(k+1)(k)x^k - \sum_{k=0}^{\infty} a_k x^k = 0$
$a_2(2)(1)x^0 + \sum_{k=1}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} a_k(k)(k-1)x^{k-1} - a_0x^0 - \sum_{k=0}^{\infty} a_k x^k = 0$
$\Rightarrow 2a_2 - a_0 + \sum_{k=1}^{\infty}[a_{k+2}(k+2)(k+1) + a_{k+1}(k+1)(k) - a_k]x^k = 0$
$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$
$a_{k+2}(k+2)(k+1) + a_{k+1}(k+1)(k) - a_k = 0$
$\Rightarrow a_{k+2} = \frac{a_k - a_{k+1}(k+1)(k)}{(k+2)(k+1)}, k \geq 1$

Thus we have,
$k = 1 \Rightarrow a_3 = \frac{a_1 - a_2(2)}{(3)(2)} = \frac{a_1 - a_0}{6}$
$k = 2 \Rightarrow a_4 = \frac{a_2 - a_3(3)(2)}{(4)(3)} = \frac{\frac{a_0}{2} - a_1 + a_0}{12} = \frac{3\frac{a_0}{2} - a_1}{12}$
$k = 3 \Rightarrow a_5 = \frac{a_3 - a_4(4)(3)}{(5)(4)} = \frac{a_1 - a_0 - 9a_0 + 6a_1}{20} = \frac{7a_1 - 10a_0}{20}$

Substituting these expressions for the coefficients into the solution
$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
yields
$y(x) = a_0 + a_1x + \frac{a_0}{2}x^2 + \frac{a_1 - a_0}{6}x^3 + \ldots$
$= a_0[1 + \frac{x^2}{2} - \frac{x^3}{6} + \ldots] \ + a_1[x + \frac{x^3}{6} + \ldots]$
$y(0) = a_0(1) = 0 \Rightarrow a_0 = 0$
$y'(0) = a_1 = 1$
$y(x) = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{7}{20}x^5 + \ldots$