MA 221 Homework Solutions
Due date: March 26/27, 2009

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Problems 1, 3, 5, 6, 13, 15, 17 and 19
(Underlined problems are to be handed in)
In problems 1, 3, 5 and 6 determine the solutions, if any, to the given boundary problem.
1.) $y'' + 2y' + 26y = 0$
$y(0) = 1, y(\pi) = -e^{-\pi}$

$$r^2 + 2r + 26 = 0$$
$$r = -1 \pm 5i$$

Therefore

$$y(x) = c_1 e^{-x} \cos 5x + c_2 e^{-x} \sin 5x$$

The BCs imply

$$y(0) = c_1 = 1$$
$$y(\pi) = c_1 e^{-\pi} \cos 5\pi = -e^{-\pi}$$
$$-c_1 e^{-\pi} = -e^{-\pi}$$
$$c_1 = 1$$

Thus

$$y(x) = e^{-x} \cos 5x + c_2 e^{-x} \sin 5x$$

Where C_2 is arbitrary

3.) $y'' - 4y' + 13y = 0$
$y(0) = 0, y(\pi) = 0$

$$r^2 - 4r + 13 = 0$$
$$r = 2 \pm 3i$$

Therefore

$$y(x) = c_1 e^{2t} \sin 3t + c_2 e^{2t} \cos 3t$$

The BCs imply

$$y(0) = c_2 = 0$$
$$y(\pi) = c_1 e^{2\pi} \sin 3\pi = 0$$

For all c_1. Thus

$$y = c_1 e^{2t} \sin 3t$$

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$$y(0) = c_2 = 0$$
$$y(\pi) = c_1 e^{2\pi} \sin 3\pi = 0$$

For all c_1. Thus

$$y = c_1 e^{2t} \sin 3t$$

3.) $y'' - 4y' + 13y = 0$
$y(0) = 0, y(\pi) = 0$

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Therefore

$$y(x) = c_1 e^{2t} \sin 3t + c_2 e^{2t} \cos 3t$$

The BCs imply

$$y(0) = c_2 = 0$$
$$y(\pi) = c_1 e^{2\pi} \sin 3\pi = 0$$

For all c_1. Thus

$$y = c_1 e^{2t} \sin 3t$$

5.) $y'' + y = \sin 2x \qquad y(0) = y(2\pi) \qquad y'(0) = y'(2\pi)$

$$r^2+1=0$$
$$r=\pm i$$
$$\alpha=0,\beta=1$$
$$y_h(t)=c_1e^0\cos x+c_2e^0\sin x$$
$$y_h(t)=c_1\cos x+c_2\sin x$$
$$y_p=A\sin 2x+B\cos 2x$$
$$y_p'=2A\cos 2x-2B\sin 2x$$
$$y_p''=-4A\sin 2x-4B\cos 2x$$
$$-4A\sin 2x-4B\cos 2x+A\sin 2x+B\cos 2x=\sin 2x$$
$$-3A\sin 2x-3B\cos 2x=\sin 2x$$
$$-3A=1,-3B=0$$
$$A=-1/3,B=0$$
$$y_p=(-1/3)\sin 2x$$
$$y(x)=c_1\cos x+c_2\sin x-(1/3)\sin 2x$$

where c_1 and c_2 are the real numbers

$y(0)=c_1$ and $y(2\pi)=c_1$ so the condition $y(0)=y(\pi)$ implies nothing.

$$y'(x)=-c_1\sin x+c_2\cos x-\frac{2}{3}\cos 2x$$

Thus

$y'(0)=c_2-\frac{2}{3}$ and $y'(2\pi)=c_2-\frac{2}{3}$ so the condition $y'(0)=y'(2\pi)$ also implies nothing.

Thus

$$y(x)=c_1\cos x+c_2\sin x-(1/3)\sin 2x$$

6.) $y''-y=x$
$y(0)=3,y'(1)=2e-e^{-1}-1$

$$r^2-1=0$$
$$r=\pm 1$$
$$y_h=c_1e^x+c_2e^{-x}$$
$$y_p=Ax+B$$
$$y_p'=A$$
$$y_p''=0$$
$$0-Ax-B=x$$
$$A=-1,B=0$$
$$y_p=-x$$

Substituting

$$y(0) = 3, y'(1) = 2e - e^{-1} - 1$$
$$c_1 = 2, c_2 = 1$$
$$y = 2e^x + e^{-x} - x$$

In Problems 13, 15, 17 and 19, find all the real eigenvalues and eigenfunctions for the given eigenvalue problem.

13.) $y'' + \lambda y = 0;$

$y(0) = 0, \quad y'(1) = 0$

The auxiliary equation for this problem is: $r^2 + \lambda = 0$.

To find eigenvalues that yield nontrivial solutions we will consider the three cases

$\lambda < 0$

$\lambda = 0$

$\lambda > 0$

Case 1: $\lambda < 0$ Let $\lambda = -\alpha^2$, where $\alpha \neq 0$. The DE becomes

$$y'' - \alpha^2 y = 0$$

In this case, the roots to the auxiliary equation are $\pm\alpha$ Therefore, a general solution to the differential equation is given by:

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

By applying the BC's:

$y(0) = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$

Thus

$$y(x) = c_1(e^{\alpha x} - e^{-\alpha x})$$

In order to apply the second BC, we need to find $y'(x)$. Thus we have:

$$y'(x) = c_1\alpha(e^{\alpha x} + e^{-\alpha x})$$

Plugging in the second BC $y'(1) = 0$

$$y'(1) = c_1\alpha(e^{\alpha} + e^{-\alpha}) = 0$$

Since $e^{\alpha} + e^{\alpha} \neq 0$, the only way the equation above can be true is for $c_1 = 0$. So in this case we have only the trivial solution. Thus, there are no eigenvalues for $\lambda < 0$.

Case2: $\lambda = 0$

In this case we are solving the differential equation $y'' = 0$. This equation has a general solution given by:

$y(x) = c_1 + c_2 x \quad \Rightarrow \quad y'(x) = c_2$

By applying the boundary conditions, we obtain

$y(0) = c_1 = 0;$

$y'(1) = c_2 = 0$

Thus, $c_1 = c_2 = 0$, and zero is not an eigenvalue

Case 3: $\lambda > 0$ Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$$y'' + \beta^2 y = 0$$

In this case the roots to the associated auxiliary equation are $r = \pm\beta i$
Therefore, the general solution is given by

$$y(x) = c_1\cos\beta x + c_2\sin\beta x$$

By applying the first boundary condition, we obtain
$y(0) = c_1 = 0 \qquad \Rightarrow$

$$y(x) = c_2\sin\beta x$$

In order to apply the second BC we need to find $y'(x)$. Thus,

$$y'(x) = c_2\beta\cos\beta x$$

Pluging in the BC

$$y'(1) = c_2\beta\cos\beta = 0$$

Therefore, in order to obtain a solution other that the trivial solution, we must have
$\cos\beta = 0 \qquad \Rightarrow \qquad \beta = \left(n + \frac{1}{2}\right)\pi, \qquad n = 0,1,2\ldots$
$\Rightarrow \lambda_n = \beta^2 = \left(n + \frac{1}{2}\right)^2\pi^2, \qquad \text{with } n = 0,1,2\ldots$
For these eigenvalues λ_n, we have the corresponding eigenfunctions,

$$y_n(x) = c_n\sin\left[\left(n + \frac{1}{2}\right)\pi x\right] \qquad \text{with } n = 0,1,2\ldots.$$

where c_n is an arbitrary nonzero constant.
15.) $y'' + 3y + \lambda y = 0;$
$y'(0) = 0, \qquad y'(\pi) = 0$
The auxiliary equation for this problem is: $r^2 + (\lambda + 3) = 0$ and $r = \pm\sqrt{-(3+\lambda)}$
To find the eigenvalues which yield nontrivial solutions, three cases must be considered:
$\lambda + 3 < 0$
$\lambda + 3 = 0$
$\lambda + 3 > 0$
Case1: $\lambda + 3 < 0$ Let $\lambda + 3 = -\alpha^2$ where $\alpha \neq 0$
In this case the roots to the auxiliary equation are the real numbers $\pm\sqrt{-(\lambda+3)}$
The general solution to $y'' - \alpha^2 y = 0$ is $y(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}$
By applying the first boundary condition, we obtain:
$y'(x) = c_1\alpha e^{\alpha x} - c_2\alpha e^{-\alpha x} = \alpha(c_1e^{\sqrt{-(\lambda+3)}\,x} - c_2e^{-\sqrt{-(\lambda+3)}\,x})$
$y'(0) = \alpha(c_1 - c_2) = 0 \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$
$y'(\pi) = \alpha(c_1e^{\alpha\pi} - c_1e^{-\alpha\pi}) = \alpha c_1(e^{\alpha\pi} - e^{-\alpha\pi}) = 0;$ since $\alpha \neq 0$ and $e^{\alpha\pi} - e^{-\alpha\pi} \neq 0$
$\Rightarrow c_1 = c_2 = 0$
In this case, we have only the trivial solution. There are no eigenvalues for $\lambda + 3 < 0$.
Case 2: $\lambda + 3 = 0$
In this case we are solving the differential equation $y'' = 0$. This equation has a general solution given by:
$y(x) = c_1 + c_2x$
$y'(x) = c_2$
By applying the boundary conditions, we obtain
$y'(0) = c_2 = 0;$
$y'(\pi) = c_2 = 0$
Thus, c_1 is arbitary and zero is an eigenvalue with eigenfunction $y(x) = C,\ C$ any constant.

Case 3: $\lambda + 3 > 0$ Let $\lambda + 3 = \beta^2$ where $\beta \neq 0$

The DE becomes

$$y'' + \beta^2 y = 0$$

Therefore, the general solution is given by

$$y(x) = c_1 \cos \beta x + c_2 \sin \beta x$$

By applying the first boundary condition, we obtain

$y'(x) = \beta(-c_1 \sin \beta x + c_2 \cos \beta x)$

$y'(0) = \beta c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow y'(x) = \beta(-c_1 \sin \beta x)$

$y'(\pi) = \beta(-c_1 \sin(\beta\pi) = 0,$ Since $\beta \neq 0$ and we want

$c_1 \neq 0 \Rightarrow \beta = n \Rightarrow \lambda + 3 = \beta^2 = n^2 \Rightarrow \lambda = n^2 - 3$

$\Rightarrow \lambda_n = n^2 - 3$ with $n = 0, 1, 2\ldots$

For these eigenvalues λ_n, we have the corresponding eigenfunctions,

$y_n(x) = c_n \cos nx$ with $n = 0, 1, 2.\ldots.$ where c_n is an arbitrary nonzero constant.

17.) $y'' + \lambda y = 0$ $2y(0) + y'(0) = 0$ $y(\pi) = 0$

The find the eigenvalues which yield nontrivial solutions, three cases must be considered:

$\lambda < 0$

$\lambda = 0$

$\lambda > 0$

Case 1: $\lambda < 0$

$\lambda = -\mu^2$ $\mu > 0$ and the DE is $y'' - \mu^2 y = 0$

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

For convenience we introduce the hyperbolic sine and cosine

$$\cosh(\mu x) = \frac{e^{\mu x} + e^{-\mu x}}{2}$$

$$\sinh(\mu x) = \frac{e^{\mu x} - e^{-\mu x}}{2}$$

and write the solution above in terms of these functions. Then

$$\begin{aligned} y(x) &= c_1(\cosh(\mu x) + \sinh(\mu x)) + c_2(\cosh(\mu x) - \sinh(\mu x)) \\ &= (c_1 + c_2)\cosh(\mu x) + (c_1 - c_2)\sinh(\mu x) \end{aligned}$$

Let

$$k_1 = (c_1 + c_2) \qquad k_2 = (c_1 - c_2)$$

then:

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} = k_1 \cosh(\mu x) + k_2 \sinh(\mu x)$$

$$y'(x) = \mu k_1 \sinh(\mu x) + \mu k_2 \cosh(\mu x)$$

$$2y(0) + y'(0) = 2k_1 + \mu k_2 = 0 \qquad \Rightarrow k_2 = \frac{-2k_1}{\mu}$$

so

$$y(x) = k_1\left(\cosh(\mu x) - \left(\frac{2}{\mu}\right)\sinh(\mu x)\right)$$

$$y(\pi) = k_1\left(\cosh(\mu\pi) - \left(\frac{2}{\mu}\right)\sinh(\mu\pi)\right) = 0$$

Since we want $k_1 \neq 0.$ then we must have

$$\frac{\mu}{2} = \tanh(\mu\pi)$$

so

$$\mu = 2\tanh(\mu\pi) \qquad \Rightarrow \qquad \lambda = -\mu^2 = -4\tanh^2(\mu\pi)$$

and

$$y(x) = k_1\left(\cosh(\mu x) - \left(\frac{2}{\mu}\right)\sinh(\mu x)\right)$$

Case 2: $\lambda = 0$
The DE becomes $y''(x) = 0,$ so $y(x) = ax + b \qquad y'(x) = a$
$2y(0) + y'(0) = 2b + a = 0 \qquad y(\pi) = a\pi + b = 0$
Thus $a = b = 0$ and we have only the trivial solution.

Case 3: $\lambda > 0$ Let $\lambda = \mu^2,$ where $\mu \neq 0.$ Then the DE becomes $y'' + \mu^2 y = 0$ and

$$y(x) = c_1\sin\mu x + c_2\cos\mu x$$

$$y'(x) = c_1\mu\cos\mu x - c_2\mu\sin x$$

$$2y(0) + y'(0) = 2c_2 + \mu c_1 = 0$$

$$y(\pi) = c_1\sin\mu\pi + c_2\cos\mu\pi = 0$$

Then:

$$c_2 = \frac{-\mu}{2}c_1$$

and

$$c_1\left(\sin\mu\pi - \frac{\mu}{2}\cos\mu\pi\right) = 0$$

so

$$\tanh\mu\pi = \frac{\mu}{2} \Rightarrow \mu = 2\tan\mu\pi$$

and

$$y = c\left(\sin\mu x - \frac{\mu}{2}\cos\mu x\right)$$

19.) $(xy')' + \lambda x^{-1} = 0 \qquad y'(0) = 0 \qquad y(e^{\pi}) = 0$
By the Cauchy-Euler equation
$(xy')' + \lambda x^{-1} = x^2y'' + xy' + \lambda y = 0 \qquad x > 0$
Substituting $y = x^r$ gives $r^2 + \lambda = 0$ as the auxiliary equation for $x^2y'' + xy' + \lambda y = 0$
Case 1: $\lambda < 0$: Let $\lambda = -\mu^2$ for $\mu > 0.$ The roots are $r = \pm\mu$
The general solution is: $y(x) = c_1x^{\mu} + c_2x^{-\mu}$
and $y'(x) = c_1\mu x^{\mu} - c_2\mu x^{-\mu-1} = \mu(c_1x^{\mu} - c_2x^{-\mu-1})$
Substituting into the first boundary condition gives
$y'(1) = \mu(c_1 - c_2) = 0$

Since $\mu > 0$

$c_1 - c_2 = 0 \quad \Rightarrow \quad c_1 = c_2 \quad \Rightarrow y(x) = c_1(x^{\mu} + x^{-\mu})$

Substituting this into the second condition yields:

$y(e^{\pi}) = c_1(e^{\mu\pi} + e^{-\mu\pi}) = 0$

Since $e^{\mu\pi} + e^{-\mu\pi} \neq 0$ the only way equation $c_1(e^{\mu\pi} + e^{-\mu\pi}) = 0$ can be true is for $c_1 = 0$.

In this case, we only have trivial solutions.

Case 2: $\lambda = 0$

In this case we are solving the differential equation $(xy')' = 0$. This equation can be solved as follows:

$xy' = c_1 \quad \Rightarrow y' = \frac{c_1}{x} \quad \Rightarrow y(x) = c_2 + c_1 \ln x$

By applying the boundary conditions, we obtain

$y'(1) = c_1 = 0 \quad y(e^{\pi}) = c_2 + c_1 \ln(e^{\pi}) = c_2 + c_1\pi = 0$

Solving these equations simultaneously yields $c_1 = c_2 = 0$. This, we again find only the trivial solution. Therefore, $\lambda = 0$ is not an eigenvalue.

Case 3: $\lambda > 0$

Let $\lambda = \mu^2$ for $\mu > 0$. The roots of the auxiliary equation are $r \pm \mu i$

The general solution is:

$y(x) = c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)$

$y'(x) = -c_1(\frac{\mu}{x}) \sin(\mu \ln x) + c_2(\frac{\mu}{x}) \cos(\mu \ln x)$

By applying the first boundary condition, we obtain

$y'(1) = c_2\mu = 0 \quad c_2 = 0$

Applying the second boundary condition, we obtain

$y(e^{\pi}) = c_1 \cos(\mu \ln(e^{\pi})) = c_1 \cos(\mu\pi) = 0$

Therefore, in order to obtain a solution other than the trivial solution, we must have

$\cos(\mu\pi) = 0 \quad \Rightarrow \quad \mu\pi = (n + \frac{1}{2})\pi \quad n = 0, 1, 2.\ldots\ldots.$

$\Rightarrow \mu = n + \frac{1}{2} \quad \Rightarrow \lambda_n = (n + \frac{1}{2})^2 \quad n = 0, 1, 2.\ldots\ldots.$

Corresponding to the eigenvalues, λ_n's, we have the eigenfunctions:

$y_n(x) = c_n \cos[(n + \frac{1}{2}) \ln x] \quad n = 0, 1, 2.\ldots..$

Where c_n is an arbitrary nonzero constant.