MA 221 Homework Solutions
Due date: March 31, 2009

10.4, pg. 611 # 5, 6, 7, 9, 11, 13, 15
(Underlined Problems are to be handed in)

In problems 5, 6, 7 and 9, compute the Fourier sine series for the given function

5.) $f(x) = -1$
$0 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{T} \text{ where } b_n = \frac{2}{T}\int_0^T f(x)\sin\frac{n\pi x}{T}dx$$

$$b_n = 2\int_0^1 -1\sin n\pi x dx = \frac{2}{n\pi}(\cos n\pi - 1)$$

Substituting

$$f(x) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin[(2k-1)]\pi x$$

6.) $f(x) = \cos x$
$0 < x < \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{T} \text{ where } b_n = \frac{2}{T}\int_0^T f(x)\sin\frac{n\pi x}{T}dx$$

$$b_n = \frac{2}{\pi n^2}\int_0^{n\pi}\cos u \sin u dx$$

Substituting

$$f(x) = \sum_{k=1}^{\infty}\frac{8k}{\pi(4k^2-1)}\sin 2kx$$

7.) $f(x) = x^2$
$0 < x < \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx dx$$

$$\frac{\pi}{2} b_n = \int_0^{\pi} x^2 \sin nx dx = [-x^2 \frac{\cos nx}{n}]_0^{\pi} + \frac{2}{n} \int_0^{\pi} x \cos nx dx$$

$$= -\frac{\pi^2 \cos nx}{n} + 0 + \frac{2}{n}[(x \frac{\sin nx}{n})_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx dx]$$

$$= -\frac{\pi^2 \cos nx}{n} + \frac{2}{n^3}(\cos n\pi - \cos 0)$$

$$b_n = \frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3}$$

Substituting

$$f(x) = x^2 = \sum_{n=1}^{\infty} \{ \frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3} \} \sin nx$$

9.) $f(x) = x - x^2$
$0 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{T} \text{ where } b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{n\pi x}{T} dx$$

$$b_n = 2 \int_0^1 (x - x^2) \sin n\pi x dx = -2 \frac{n\pi \sin n\pi + 2\cos n\pi - 2}{n^3 \pi^3}$$

Substituting

$$f(x) = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3 \pi^3} \sin(2k+1)\pi x$$

In problems 11,13 and 15, compute the Fourier cosine series for the given function
11.) $f(x) = \pi - x$
$0 < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{T} \quad \text{where } a_n = \frac{2}{T} \int_0^T f(x) \cos \frac{n\pi x}{T} dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx = -\frac{2}{\pi} \frac{\cos n\pi - 1}{n^2} \quad n \geq 1$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \pi$$

Substituting

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x$$

13.) $f(x) = e^x$
$0 < x < 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T} \quad \text{where } a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}\,dx$$

$$a_0 = 2\int_0^1 e^x \cos nx dx = -2(e-1) \text{ (since } \cos 0 = 1)$$

Integrating by parts:

$$2\int_0^1 e^x \cos nx dx = \frac{e^x(\cos n\pi x + n\pi \sin n\pi x)}{1 + n^2\pi^2}$$

$$a_n = 2\int_0^1 e^x \cos nx dx = \left[\frac{2e^x(\cos n\pi x + n\pi \sin n\pi x)}{1 + n^2\pi^2}\right]_0^2 = \frac{2[(-1)^n e - 1]}{1 + n^2\pi^2}$$

$$f(x) = e - 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n e - 1}{1 + n^2\pi^2} \cos n\pi x$$

15.) $f(x) = \sin x$

$0 < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T} \quad \text{where } a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx = -\frac{2}{\pi}\,\frac{\cos n\pi + 1}{-1 + n^2}$$

Substituting

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right)\cos 2kx$$

$$\int \sin x \cos nx dx = \tfrac{1}{2n^2-2}(\cos(x+nx) - n\cos(x+nx) + \cos(x-nx) + n\cos(x-nx))$$