MA 221 Homework Solutions
Due date: March 5/6, 2009

7.5 p. 383 #1, 3, 5, 6, 7, 15, 17, 19

(Underlined problems are to be handed in)

For problems 1, 3, 5, and 6, solve the given initial value problem using the method of Laplace transforms.

1.) $y'' - 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 4$

Taking the Laplace transform of $y'' - 2y' + 5y = 0$ and applying the linearity of the Laplace transform yields:

$L\{y''\} - 2L\{y'\} + 5L\{y\} = L\{0\}$

If we put $Y(s) = L\{y\}(s)$ and apply properties (2) and (4) on pages 361 and 362 of the text, we get:

$L\{y'\} = sY(s) - y(0) = sY(s) - 2$

and

$L\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4$

Substituting these expressions and using the fact that $L\{0\} = 0$ and solving for $Y(s)$ yields

$[s^2Y(s) - 2s - 4] - 2[sY(s) - 2] + 5Y(s) = 0$

Solving for $Y(s)$ gives

$Y(s) = \frac{2s}{s^2-2s+5} = \frac{2s}{(s^2-2s+1)+4} = \frac{2s}{(s-1)^2+2^2} = 2(\frac{s}{(s-1)^2+2^2}) = 2(\frac{s-1+1}{(s-1)^2+2^2}) = 2(\frac{s-1}{(s-1)^2+2^2} + \frac{1}{(s-1)^2+2^2})$

Using Table 7.1 on page 358 of the text to find the inverse Laplace transform of the above gives:

$y(t) = 2e^t \cos 2t + e^t \sin 2t$

5.) $w'' + w = t^2 + 2; \quad w(0) = 1, \quad w'(0) = -1$

$\mathcal{L}\{w''\}(s) + W(s) = \mathcal{L}\{t^2 + 2\}(s) = \mathcal{L}\{t^2\}(s) + 2\mathcal{L}\{1\}(s) = \frac{2}{s^3} + \frac{2}{s}$

Since $\mathcal{L}\{w''\}(s) = s^2W(s) - sw(0) - w'(0) = s^2W(s) - s + 1,$ we have

$[s^2W(s) - s + 1] + W(s) = \frac{2}{s^3} + \frac{2}{s} \Rightarrow (s^2 + 1)W(s) = s - 1 + \frac{2(s^2+1)}{s^3} \Rightarrow W(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1}$

Now taking the inverse Laplace transform, we obtain

$w = \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} - \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} + \mathcal{L}^{-1}\{\frac{2}{s^3}\} = \cos t - \sin t + t^2$

6.) $y'' - 4y' + 5y = 4e^{3t}; \quad y(0) = 2, \quad y'(0) = 7$

Let $Y = L\{y\}$. Then taking the Laplace transform of the equation and using linearity yields

$L\{y''\} - 4L\{y']\} + 5L\{y\} = L\{4e^{3t}\}$

Solving for Y gives

$[s^2Y - 2s - 7] - 4[sY - 2] + 5Y = \frac{4}{s-3}$

$\rightarrow \quad (s^2 - 4s + 5)Y = 2s - 1 + \frac{4}{s-3}$

$\rightarrow \quad Y = \frac{2s-1+\frac{4}{s-3}}{s^2-4s+5} = \frac{2s^2-7s+7}{(s-3)(s^2-4s+5)} = \frac{2}{s-3} + \frac{1}{s^2-4s+5} = \frac{2}{s-3} + \frac{1}{s^2-4s+4+1} = \frac{2}{s-3} + \frac{1}{(s-2)^2+1}$

Now, taking the inverse Laplace transform, we obtain

$y = L^{-1}\{\frac{2}{s-3}\} + L^{-1}\{\frac{1}{(s-2)^2+1}\}$

$y = 2e^{3t} + e^{2t}\sin t$

For problems 15 and 19, solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given intial value problem.

15.) Solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem.

$y'' - 3y' + 2y = \cos t; \qquad y(0) = 0, \qquad y'(0) = -1$

Taking the Laplace transform and applying the linearity of the Laplace transform yields

$L\{y''\} - 3L\{y'\} + 2L\{y\} = L\{\cos t\}$

If we put $Y(s) = L\{y\}(s)$, we get

$L\{y'\} = sY(s)$

and

$L\{y''\} = s^2Y(s) + 1$

Combining, and knowing the fact that $L\{\cos t\} = \frac{s}{s^2+1}$, yields the equation

$s^2Y(s) + 1 - 3sY(s) + 2Y(s) = \frac{s}{s^2+1}$

Solving for $Y(s)$, we get

$Y(s) = \frac{-s^2+s-1}{(s^2+1)(s^2-3s+2)} = \frac{-s^2+s-1}{(s^2+1)(s-1)(s-2)}$

19.) $y'' + 5y' - y = e^t - 1$

$y(0) = 1, y'(0) = 1$

$Y(s) := \mathcal{L}\{y\}(s)$

$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1, \qquad \mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s -$

The Laplace transform applied to both sides of the equation, yields

$[s^2Y(s) - s - 1] + 5[sY(s) - 1] - Y(s) = \mathcal{L}\{e^t\}(s) - \mathcal{L}\{1\}(s) = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)} \Rightarrow$

$(s^2 + 5s - 1)Y(s) = \frac{1}{s(s-1)} + s + 6 = \frac{s^3+5s^2-6s+1}{s(s-1)} \Rightarrow \qquad Y(s) = \frac{s^3+5s^2-6s+1}{s(s-1)(s^2+5s-1)}$