MA 221 Homework Solutions
Due date: April 2/3, 2009

Sec 10.2 Problems #19, 20, 22
Sec 10.6 Problem #3

Section 10.2

Solve the vibrating string problem with $\alpha = 3$, $L = \pi$, and the given initial functions $f(x)$ and $g(x)$.

19.) $f(x) = 3\sin 2x + 12\sin 13x, \qquad g(x) = 0$

By letting $\alpha = 3$ and $L = \pi$ in formula (24) on page 585 of the text, we see that the solution we want will have the form

$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos 3nt + b_n \sin 3nt]\sin nx$

Therefore, we see that

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}[-3na_n \sin 3nt + 3nb_n \cos 3nt]\sin nx$

In order for the solution to satisfy the initial conditions, we must find a_n and b_n such that

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = 3\sin 2x + 12\sin 13x, \qquad (1)$

and

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} 3nb_n \sin nx = 0.$

From the first condition, we observe that we must have a term for $n = 2, 13$ and for these terms we want $a_2 = 3$ and $a_{13} = 12$.

All of the other a_n's must be zero. By comparing coefficients in the second condition, we see that all b_n's must also be zero.

Therefore, by substituting these values into equation (1) above, we obtain the solution of the vibrating string problem with

$\alpha = 3$, $L = \pi$ and $f(x)$ and $g(x)$ as given. This solution is given by

$u(x,t) = 3[\cos(3)(2)t]\sin 2x + 12[\cos(3)(13)t]\sin 13x + 0$

Or by simplifying, we obtain

$u(x,t) = 3\cos 6t \sin 2x + 12\cos 39t \sin 13x$

20.) $f(x) = 0, \qquad g(x) = -2\sin 3x + 9\sin 7x - \sin 10x$

By letting $\alpha = 3$ and $L = \pi$ in formula (24) on page 585 of the text, we see that the solution we want will have the form

$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos 3nt + b_n \sin 3nt]\sin nx$

Therefore, we see that

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}[-3na_n \sin 3nt + 3nb_n \cos 3nt]\sin nx$

In order for the solution to satisfy the initial conditions, we must find a_n and b_n such that

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = 0$

and

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} 3nb_n \sin nx = -2\sin 3x + 9\sin 7x - \sin 10x \qquad (2)$

From the first condition, we observe that $a_n = 0$

By comparing coefficients in the second condition, we see that we require

$9b_3 = -2$ or $b_3 = -\frac{2}{9}$, $\qquad 21b_7 = 9$ or $b_7 = \frac{3}{7}$

$30b_{10} = -2$ or $b_{10} = -\frac{1}{30}$ and all other b_n values must be zero. Therefore, by substituting these values into equation (2) above, we obtain the

solution of the vibrating string problem with $\alpha = 3$, $L = \pi$ and $f(x)$ and $g(x)$ as given. This solution is given by

$$\boxed{u(x,t) = -\frac{2}{9}\sin 9t \sin 3x + \frac{3}{7}\sin 21t \sin 7x - \frac{1}{30} 30t \sin 10x}$$

22.)

$f(x) = \sin x - \sin 2x + \sin 3x, \qquad g(x) = 6\sin 3x - 7\sin 5x$

By letting $\alpha = 3$ and $L = \pi$ in formula (24) on page 585 of the text, we see that the solution we want will have the form

$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos 3nt + b_n \sin 3nt]\sin nx$

Therefore, we see that

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}[-3na_n \sin 3nt + 3nb_n \cos 3nt]\sin nx$

In order for the solution to satisfy the initial conditions, we must find a_n and b_n such that

$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = \sin x - \sin 2x + \sin 3x$

and

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} 3nb_n \sin nx = 6\sin 3x - 7\sin 5x \qquad (2)$

From the first condition, we observe that we must have a term for $n = 1,2,3$ and for these terms we want $a_1 = 1, a_2 = -1$ and $a_3 = 1$.

All of the other a_n's must be zero. By comparing coefficients in the second condition, we see that we require

$(3)(3)b_3 = 6$ or $b_3 = \frac{2}{3}$, $\qquad (3)(5)b_5 = -7$ or $b_5 = -\frac{7}{15}$

We also see that all other values for b_n must be zero. Therefore, by substituting these values into equation (2) above, we obtain the

solution of the vibrating string problem with $\alpha = 3$, $L = \pi$ and $f(x)$ and $g(x)$ as given. This solution is given by

$$\boxed{u(x,t) = \cos 3t \sin x - \cos 6t \sin 2x + \cos 9t \sin 3x + \frac{2}{3}\sin 9t \sin 3x - \frac{7}{15}\sin 15t \sin 5x}$$

Section 10.6 Problem 3

Find a formal solution to the vibrating string problem governed by the given initial-boundary value problem.

3.) $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \qquad t > 0,$

$u(t,0) = u(\pi,t) = 0, \qquad t > 0,$

$u(x,0) = x^2(\pi - x), \qquad 0 < x < \pi,$

$\dfrac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < \pi$

This problem has the form of the problem given in equations (1) - (4) on page 625of the text. Here, however, $\alpha = 2$, $L = \pi$,

and $f(x) = x^2(\pi - x)$, and $g(x) = 0$. This problem is consistent because

$f(0) = 0 = f(\pi)$, and $g(0) = 0 = g(\pi)$

The solution to this problem was derived in Section 10.2 of the text and given again in equation (5) on page 625 of the text.

Making appropriate substitutions in equation (5) yields a formal solution given by

$u(x,t) = \sum_{n=1}^{\infty}[a_n \cos n\pi t + b_n \sin n\pi t]\sin n\pi x$

To find the a_n's we note that they are the Fourier sine coefficients for $x^2(\pi - x)$ and so are given by equation (7) on page 609

of the text. Thus, for $n = 1,2,3\ldots$, we have

$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi x^2 - x^3)\sin nx dx = 2\int_0^{\pi} x^2 \sin nx dx - \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx dx$

Integrating yields

$\Rightarrow \int_0^{\pi}(x^2 \sin x)dx = -\frac{1}{n}\int_0^{\pi} x^2 d(\cos nx) = -\frac{1}{n}\left(x^2\cos nx - 2\int_0^{\pi} x\cos nx dx\right)$

$= -\frac{1}{n}\left(\pi^2\cos\pi n - \frac{2}{\pi}\int_0^{\pi} x d(\sin nx)\right) = -\frac{\pi^2}{n}\cos\pi n + \frac{2}{n^2}\left(x\sin nx - \int_0^{\pi}\sin x n dx\right)$

$= -\frac{\pi^2}{n}\cos\pi n + \frac{2}{n^3\cos nx} = \frac{\pi^2}{n}\cos\pi n + \frac{2}{n^3}(\cos\pi n - 1) = \boxed{\cos\pi n\left(\frac{2}{n^3} - \frac{\pi^2}{n}\right) - \frac{2}{n^3}}$

$\Rightarrow \int_0^{\pi} x^3 \sin nx dx = -\frac{1}{n}\left[\int_0^{\pi} x^3 d(\cos nx) = -\frac{1}{n}x^3\sin nx - 3\int_0^{\pi} x^2\cos nx dx\right]$

$= -\frac{1}{n}\left[\pi^3\cos\pi n - \frac{3}{n}\int_0^{\pi} x^2 d(\sin nx)\right] = -\frac{\pi^3}{n}\cos\pi n + \frac{3}{n^2}\left[x^2\sin nx - 2\int_0^{\pi}\sin nx * x dx\right]$

$= -\frac{\pi^3}{n}\cos\pi n + \frac{6}{n^3}\int_0^{\pi} x d(\cos nx) = -\frac{\pi^3}{n}\cos\pi n + \frac{6}{n^3}(x\cos nx - \int_0^{\pi}\cos nx dx)$

$= -\frac{\pi^3}{n}\cos\pi n + \frac{6}{n^3}(\pi\cos\pi n - \frac{1}{n}\sin nx) = \boxed{-\frac{\pi^3}{n}\cos\pi n + \frac{6\pi}{n^3}\cos\pi n}$

Therefore

$a_n = 2\left(\frac{2}{n^3} - \frac{\pi^2}{n}\right)\cos\pi n - \frac{4}{n^3} - \frac{2}{\pi}\left(-\frac{\pi^3}{n} + \frac{6\pi}{n^3}\right)\cos\pi n$

$= \left(\frac{4}{n^3} - \frac{2\pi^2}{n} + \frac{2\pi^2}{n} - \frac{12}{n^3}\right)\cos\pi n - \frac{4}{n^3}$

$= -\frac{8}{n^3}\cos\pi n - \frac{4}{n^3} = -\frac{4}{n^3}(2\cos\pi n + 1)$

$\boxed{a_n = -\frac{4}{n^3}(2\cos\pi n + 1)}$

$b_n = 0;$

$u(x,t) = \sum_{n=1}^{\infty}[a_n\cos n\pi t + b_n \sin n\pi t]$

$a_n = \frac{4}{n^3}(-2\cos\pi n + 1) = \frac{4}{n^3}((-1)(-1)^n 2 - 1) = \frac{4}{n^3}(2(-1)^{n+1} - 1)$

$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{4}{n^3}(2(-1)^{n+1} - 1)\cos 2nt \sin nx}$