MA 221 Homework Solutions
Due date: April 7, 2009

Section 10.6 Problems 5, 9, 11

(Underlined problems are to be handed in)

5.) $f(x) = h_0 x/a, \qquad 0 < x \le a,$

$f(x) = h_0(L - x)/(L - a), \qquad a < x < L$

and $g(0) \equiv 0$

The problem is consistent because

$g(0) = 0 = g(L) \qquad \text{and} \qquad f(0) = 0 = f(L)$

The formal solution is given in equation (5) on page 625 of the text with the coefficients given in equations (6) and (7) on page 626. By equation (7)

$g(x) = 0 = \sum_{n=1}^{\infty} b_n(\frac{n\pi\alpha}{L})\sin(\frac{n\pi x}{L})$.

Thus, each term in this infinite series must be zero and so $b_n = 0$ for all n's. Therefore, the formal solution given in equation (5) on page 625 becomes

$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi\alpha t}{L})\sin(\frac{n\pi x}{L})$

Using equation (7) on page 609 for $n = 1, 2, 3, \ldots$ we have

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx = \frac{2}{L}[\frac{h_0}{a}\int_0^a x\sin(\frac{n\pi x}{L})dx + h_0\int_a^L \frac{L-x}{L-a}\sin(\frac{n\pi x}{L})dx]$$

$$= \frac{2h_0}{L}[\frac{1}{a}\int_0^a x\sin(\frac{n\pi x}{L})dx + \frac{L}{L-a}\int_a^L \sin(\frac{n\pi x}{L})dx - \frac{1}{L-a}\int_a^L x\sin(\frac{n\pi x}{L})dx]$$

By using integration by parts

$$\int x\sin(\frac{n\pi x}{L})dx = -\frac{xL}{n\pi}\cos(\frac{n\pi x}{L}) + \frac{L^2}{n^2\pi^2}\sin(\frac{n\pi x}{L})$$

For $n = 1, 2, 3, \ldots$

$$a_n = \frac{2h_0}{L}\{\frac{1}{a}[-\frac{aL}{n\pi}\cos(\frac{n\pi a}{L}) + \frac{L^2}{n^2\pi^2}\sin(\frac{n\pi a}{L})]$$
$$- \frac{L^2}{n\pi(L-a)}[\cos n\pi - \cos(\frac{n\pi a}{L})] - \frac{1}{L-a}[-\frac{L^2}{n\pi}\cos n\pi + \frac{aL}{n\pi}\cos(\frac{n\pi a}{L})] + \frac{L^2}{n^2\pi^2}[\sin n\pi$$

$$a_n = \frac{2h_0L^2}{n^2\pi^2 a(L-a)}\sin(\frac{n\pi a}{L}), \qquad n = 1, 2, 3, \ldots$$

$$u(x,t) = \frac{2h_0L^2}{\pi^2 a(L-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(\frac{n\pi a}{L})\cos(\frac{n\pi\alpha t}{L})\sin(\frac{n\pi x}{L})$$

9.) $\frac{\partial^2 u}{\partial t^2} = \alpha^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0$

$u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0, \qquad t > 0$

$u(x,0) = f(x), \qquad 0 < x < L$

$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L$

$X(x)T''(t) = \alpha^2 X''(x)T(t)$

$\frac{T''(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$

$\frac{T''(t)}{\alpha^2 T(t)} = K \Rightarrow T''(t) - \alpha^2 K T(t) = 0$

$\frac{X''(x)}{X(x)} = K \Rightarrow X''(x) - KX(x) = 0$

We have the BCs $X(0) = 0$ and $X'(L) = 0$

For a nontrivial solution for $X(x)$ we need sines or cosines. Thus we set $K = -\beta^2$, where $\beta \neq 0$. Then

$$X(x) = a\sin\beta x + b\cos\beta x$$

Now $X(0) = 0$ implies $b = 0$, whereas $X'(x) = a\beta\cos\beta x$ so $X'(L) = 0$ implies that $\cos\beta L = 0$. Thus

$$\beta L = \left(\frac{2n+1}{2}\right)\pi, \quad n = 0,1,2,\dots$$

or

$$\beta = \left(\frac{2n+1}{2}\right)\frac{\pi}{L}$$

and

$$X_n(x) = a_n \sin\left(\frac{2n+1}{2}\right)\left(\frac{\pi x}{L}\right), \quad n = 0,1,2,\dots$$

The equation for $T(t)$ is

$$T''(t) - \alpha^2 KT(t) = T'' + \alpha^2\left(\left(\frac{2n+1}{2}\right)\left(\frac{\pi}{L}\right)\right)^2 T = 0$$

so

$$T_n(t) = b_n \sin\left(\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha t}{L}\right)\right) + c_n \cos\left(\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha t}{L}\right)\right), \quad n = 0,1,2,\dots$$

Therefore

$$u(x,t) = \sum_{n=0}^{\infty}\left[B_n \sin\left(\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha t}{L}\right)\right) + C_n \cos\left(\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha t}{L}\right)\right)\right]\sin\left(\frac{2n+1}{2}\right)\left(\frac{\pi x}{L}\right)$$

,

The initial conditions imply

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{2n+1}{2}\right)\left(\frac{\pi x}{L}\right)$$

Also

$$u_t(x,t) = \sum_{n=0}^{\infty}\left[B_n\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha}{L}\right)\cos\left(\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha t}{L}\right)\right)\right]\sin\left(\frac{2n+1}{2}\right)\left(\frac{\pi x}{L}\right)$$

$$-\sum_{n=0}^{\infty}\left[C_n\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha}{L}\right)\sin\left(\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha t}{L}\right)\right)\right]\sin\left(\frac{2n+1}{2}\right)\left(\frac{\pi x}{L}\right)$$

Therefore

$$u_t(x,0) = g(x) = \sum_{n=0}^{\infty} B_n\left(\frac{2n+1}{2}\right)\left(\frac{\pi\alpha}{L}\right)\sin\left(\frac{2n+1}{2}\right)\left(\frac{\pi x}{L}\right)$$

11.) $\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \alpha^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0$

$u(0,t) = u(L,t) = 0, \qquad t > 0$

$u(x,0) = f(x), \qquad 0 < x < L$

$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L$

Assume the solution has the form $u(x,t) = X(x)T(t)$.

Substituting this into the partial diff. eq. yields

$X(x)T''(t) + X(x)T'(t) + X(x)T(t) = \alpha^2 X''(x)T(t)$

$\frac{T''(t)+T'(t)+T(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$

$\frac{T''(t)+T'(t)+T(t)}{\alpha^2 T(t)} = K \Rightarrow T''(t) + T'(t) + (1 - \alpha^2 K)T(t) = 0$
$\frac{X''(x)}{X(x)} = K \Rightarrow X''(x) - KX(x) = 0$
Substituting $u(x,t) = X(x)T(t)$ into the boundary conditions yields
$X(0)T(t) = 0 = X(L)T(t), \qquad t > 0$
$\Rightarrow X(0) = X(L) = 0$
$X''(x) - KX(x) = 0$ with $X(0) = X(L) = 0$
$K = -(n\pi/L)^2, \qquad n = 1,2,3,\ldots$

$$X_n(x) = A_n \sin(n\pi x/L), \qquad n = 1,2,3\ldots$$

Now, plug K into the above equation
$T''(t) + T'(t) + (1 + \frac{\alpha^2 n^2 \pi^2}{L^2})T(t) = 0, \qquad n = 1,2,3\ldots$
$r^2 + r + (1 + \frac{\alpha^2 n^2 \pi^2}{L^2}) = 0$
$r = -\frac{1}{2} \pm \frac{\sqrt{3L^2+4\alpha^2 n^2 \pi^2}}{2L} i, \qquad n = 1,2,3\ldots$
$T_n(t) = e^{-t/2}[B_n \cos(\frac{\sqrt{3L^2+4\alpha^2 n^2 \pi^2}}{2L} t) + C_n \sin(\frac{\sqrt{3L^2+4\alpha^2 n^2 \pi^2}}{2L} t)]$
Let $\beta_n = \frac{\sqrt{3L^2+4\alpha^2 n^2 \pi^2}}{2L}$ tp get

$$T_n(t) = e^{-t/2}[B_n \cos(\beta_n t) + C_n \sin(\beta_n t)]$$

$$u_n(x,t) = X_n(x)T_n(t) = A_n e^{-t/2}[B_n \cos(\beta_n t) + C_n \sin(\beta_n t)] \sin(n\pi x/L)$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-t/2}[a_n \cos(\beta_n t) + b_n \sin(\beta_n t)] \sin(n\pi x/L)$$

where, $a_n = A_n B_n$ and $b_n = A_n C_n$
Since

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} \{(-1/2)e^{-t/2}[a_n \cos(\beta_n t) + b_n \sin(\beta_n t)] + e^{-t/2}[-a_n \beta_n \sin \beta_n t + b_n \beta_n \cos(\beta_n t)]\} \sin(n\pi x/L)$$

$$\frac{\partial u(x,0)}{\partial t} = 0 = \sum_{n=1}^{\infty} \{-\frac{a_n}{2} + b_n \beta_n\} \sin(n\pi x/L)$$

$\Rightarrow$

$$-\frac{a_n}{2} + b_n \beta_n = 0 \Rightarrow b_n = \frac{a_n}{2\beta_n}, \qquad n = 1,2,3\ldots$$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-t/2}[\cos(\beta_n t) + \frac{1}{2\beta_n} \sin(\beta_n t)] \sin(n\pi x/L)$$

To find a_n use the remaining initial condition to obtain
$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$
Therefore,
$a_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) dx$
Formal solution to the problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-t/2}[\cos(\beta_n t) + \frac{1}{2\beta_n}\sin(\beta_n t)]\sin(n\pi x/L)$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

and

$$\beta_n = \frac{\sqrt{3L^2 + 4\alpha^2 n^2 \pi^2}}{2L}$$