SOLUTION Here $f(x, y) = 3y^{2/3}$ and $\partial f/\partial y = 2y^{-1/3}$. Unfortunately $\partial f/\partial y$ is not continuous or even defined when $y = 0$. Consequently, there is no rectangle containing $(2, 0)$ in which both f and $\partial f/\partial y$ are continuous. Because the hypotheses of Theorem 1 do not hold, we cannot use Theorem 1 to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution. We refer you to Problem 29 for the details. ■

In Example 9 suppose the initial condition is changed to $y(2) = 1$. Then, since f and $\partial f/\partial y$ are continuous in any rectangle that contains the point $(2, 1)$ but does not intersect the x-axis—say, $R = \{(x, y): 0 < x < 10, 0 < y < 5\}$—it follows from Theorem 1 that this *new* initial value problem has a unique solution in some interval about $x = 2$.

EXERCISES 1.2

1. **(a)** Show that $\phi(x) = 2x^3$ is an explicit solution to

$$x\frac{dy}{dx} = 3y$$

on the interval $(-\infty, \infty)$.

(b) Show that $\phi(x) = e^{2x} - x$ is an explicit solution to

$$\frac{dy}{dx} + y^2 = e^{4x} + 2(1 - x)e^{2x} + x^2 - 1$$

on the interval $(-\infty, \infty)$.

(c) Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2 d^2y/dx^2 = 2y$ on the interval $(0, \infty)$.

2. **(a)** Show that $y^2 + x - 3 = 0$ is an implicit solution to $dy/dx = -1/(2y)$ on the interval $(-\infty, 3)$.

(b) Show that $xy^3 - xy^3 \sin x = 1$ is an implicit solution to

$$\frac{dy}{dx} = \frac{(x \cos x - \sin x - 1)y}{3(x - x \sin x)}$$

on the interval $(0, \pi/2)$.

In Problems 3–8, determine whether the given function is a solution to the given differential equation.

3. $y = \sin x + x^2$, $\quad \dfrac{d^2y}{dx^2} + y = x^2 + 2$

4. $x = 3 \cos t - 5 \sin t$, $\quad x'' + x = 0$

5. $x = \cos 2t$, $\quad \dfrac{dx}{dt} + tx = \sin 2t$

6. $\theta = 2e^{3t} - e^{2t}$, $\quad \dfrac{d^2\theta}{dt^2} - \theta\dfrac{d\theta}{dt} + 3\theta = -2e^{2t}$

7. $y = e^{2x} - 3e^{-x}$, $\quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

8. $y = 3 \sin 2x + e^{-x}$, $\quad y'' + 4y = 5e^{-x}$

In Problems 9–13, determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

9. $x^2 + y^2 = 6$, $\quad \dfrac{dy}{dx} = \dfrac{x}{y}$

10. $y - \ln y = x^2 + 1$, $\quad \dfrac{dy}{dx} = \dfrac{2xy}{y - 1}$

11. $e^{xy} + y = x - 1$, $\quad \dfrac{dy}{dx} = \dfrac{e^{-xy} - y}{e^{-xy} + x}$

12. $x^2 - \sin(x + y) = 1$, $\quad \dfrac{dy}{dx} = 2x \sec(x + y) - 1$

13. $\sin y + xy - x^3 = 2$,

$$y'' = \frac{6xy' + (y')^3 \sin y - 2(y')^2}{3x^2 - y}$$

14. Show that $\phi(x) = c_1 \sin x + c_2 \cos x$ is a solution to $d^2y/dx^2 + y = 0$ for any choice of the constants c_1 and c_2. Thus, $c_1 \sin x + c_2 \cos x$ is a two-parameter family of solutions to the differential equation.

15. Show that $\phi(x) = Ce^{3x} + 1$ is a solution to $dy/dx - 3y = -3$ for any choice of the constant C. Thus, $Ce^{3x} + 1$ is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

16. Verify that $x^2 + cy^2 = 1$, where c is an arbitrary nonzero constant, is a one-parameter family of implicit solutions to

$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$

and graph several of the solution curves using the same coordinate axes.

17. Verify that $\phi(x) = 2/(1 - ce^x)$, where c is an arbitrary constant, is a one-parameter family of solutions to

$$\frac{dy}{dx} = \frac{y(y-2)}{2} .$$

Graph the solution curves corresponding to $c = 0$, ± 1, ± 2 using the same coordinate axes.

18. Let $c > 0$. Show that the function $\phi(x) = (c^2 - x^2)^{-1}$ is a solution to the initial value problem $dy/dx = 2xy^2$, $y(0) = 1/c^2$, on the interval $-c < x < c$. Note that this solution becomes unbounded as x approaches $\pm c$. Thus, the solution exists on the interval $(-\delta, \delta)$ with $\delta = c$, but not for larger δ. This illustrates that in Theorem 1 the existence interval can be quite small (if c is small) or quite large (if c is large). Notice also that there is no clue from the equation $dy/dx = 2xy^2$ itself, or from the initial value, that the solution will "blow up" at $x = \pm c$.

19. Show that the equation $(dy/dx)^2 + y^2 + 3 = 0$ has no (real-valued) solution.

20. Determine for which values of m the function $\phi(x) = e^{mx}$ is a solution to the given equation.

(a) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = 0$

(b) $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$

21. Determine for which values of m the function $\phi(x) = x^m$ is a solution to the given equation.

(a) $3x^2\dfrac{d^2y}{dx^2} + 11x\dfrac{dy}{dx} - 3y = 0$

(b) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 5y = 0$

22. Verify that the function $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to the linear equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that each of the following initial conditions is satisfied.

(a) $y(0) = 2$, $y'(0) = 1$
(b) $y(1) = 1$, $y'(1) = 0$

In Problems 23–28, determine whether Theorem 1 implies that the given initial value problem has a unique solution.

23. $\dfrac{dy}{dx} = x^3 - y^3$, $\quad y(0) = 6$

24. $\dfrac{dy}{d\theta} - \theta y = \sin^2\theta$, $\quad y(\pi) = 5$

25. $x\dfrac{dx}{dt} + 4t = 0$, $\quad x(2) = -\pi$

26. $\dfrac{dy}{dx} = 3x - \sqrt[3]{y-1}$, $\quad y(2) = 1$

27. $y\dfrac{dy}{dx} = x$, $\quad y(1) = 0$

28. $\dfrac{dx}{dt} + \cos x = \sin t$, $\quad x(\pi) = 0$

29. **(a)** For the initial value problem (12) of Example 9, show that $\phi_1(x) \equiv 0$ and $\phi_2(x) = (x-2)^3$ are solutions. Hence, this initial value problem has multiple solutions.
(b) Does the initial value problem $y' = 3y^{2/3}$, $y(0) = 10^{-7}$, have a unique solution in a neighborhood of $x = 0$?

30. **Implicit Function Theorem.** *Let $G(x, y)$ have continuous first partial derivatives in the rectangle $R = \{(x, y): a < x < b, c < y < d\}$ containing the point (x_0, y_0). If $G(x_0, y_0) = 0$ and the partial derivative $G_y(x_0, y_0) \neq 0$, then there exists a differentiable function $y = \phi(x)$, defined in some interval $I = (x_0 - \delta, x_0 + \delta)$, that satisfies $G(x, \phi(x)) = 0$ for all $x \in I$.*

The implicit function theorem gives conditions under which the relationship $G(x, y) = 0$ defines y implicitly as a function of x. Use the implicit func-