Ma 221 Final Exam Review

Problems on Series Solutions of DEs, Fourier Series, Boundary Value Problems and Separation of Variables

Series Solutions of Differential Equations

Example

Solve the equation

$$(1-x^2)y'' + 2y = 0$$

near $x = 0$ using series solutions.

Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

Substituting into the DE we have

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=2}^{\infty} a_n (n)(n-1) x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=2}^{\infty} a_n [(n)(n-1) - 2] x^n + 2a_0 + 2a_1 x = 0$$

We now shift the first sum by letting $k = n-2$ or $n = k+2$. Then we have

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k - \sum_{n=2}^{\infty} a_n [(n)(n-1) - 2] x^n + 2a_0 + 2a_1 x = 0$$

or replacing k and n by another "dummy" variable m

$$a_2(2)(1) + a_3(3)(2)x + 2a_0 + 2a_1 x + \sum_{m=2}^{\infty} \left\{ a_{m+2}(m+2)(m+1) - a_m\left[m^2 - m - 2\right] \right\} x^m = 0$$

Therefore

$$a_2 = -a_0$$

$$a_3 = -\frac{1}{3} a_1$$

and

$$a_{m+2}(m+2)(m+1) - a_m\left[m^2 - m - 2\right] \quad m = 2, 3, \ldots$$

The recurrence relation is

$$a_{m+2} = \frac{(m-2)(m+1)}{(m+2)(m+1)} a_m = \frac{m-2}{m+2} a_m \quad m = 2, 3, \ldots$$

Hence $a_4 = 0,$ and all even coefficients $a_{2i} = 0$ for $i = 2, 3, \ldots$

$$m = 3 \Rightarrow a_5 = \frac{1}{5}a_3 = -\frac{1}{15}a_1$$

$$m = 5 \Rightarrow a_7 = \frac{3}{7}a_5 = -\frac{1}{35}a_1$$

Hence

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0\left[1 - x^2\right] + a_1\left[x - \frac{1}{3}x^3 - \frac{1}{15}x^5 - \frac{1}{35}x^7 + \cdots\right]$$

SNB Check: $(1 - x^2)y'' + 2y = 0$, Series solution is:
$y(x) = y(0) + y'(0)x - y(0)x^2 - \frac{1}{3}y'(0)x^3 - \frac{1}{15}y'(0)x^5 - \frac{1}{35}y'(0)x^7$

Example

Solve the equation

$$(x^2 + 1)y'' + xy' - y = 0$$

near $x = 0$. Be sure to give the recurrence relation and the first 5 nonzero terms of the general solution. Indicate the two linearly independent solutions.

Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

The DE implies

$$\sum_{n=2}^{\infty} a_n (n)(n-1)x^n + \sum_{n=2}^{\infty} a_n (n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n (n)x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Combining the there sums that have x^n in them, and shifting the sum with x^{n-2} in it by letting $k = n - 2$ or $n = k + 2$ we have

$$-a_0 - a_1 x + a_1 x + \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \sum_{n=2}^{\infty} a_n[(n)(n-1) + n - 1]x^n = 0$$

Or after replacing k and n by the "dummy" place keeper m

$$-a_0 + a_2(2)(1) + a_3(3)(2)x + \sum_{m=2}^{\infty}\left\{a_{m+2}(m+2)(m+1) + a_m\left(m^2 - 1\right)\right\}x^m = 0$$

Thus

$$a_2 = \frac{1}{2}a_0$$
$$a_3 = 0$$
$$a_{m+2} = -\frac{m^2-1}{(m+2)(m+1)}a_m = -\frac{m-1}{m+2}a_m \quad m = 2,3,\ldots$$
$$m = 2 \Rightarrow a_4 = -\frac{1}{4}a_2 = -\frac{1}{8}a_0$$
$$m = 3 \Rightarrow a_5 = 0$$
$$m = 4 \Rightarrow a_6 = -\frac{3}{6}a_4 = +\frac{1}{16}a_0$$

All of the odd coefficients $a_{2j+1} = 0$ for $j \geq 1$. Therefore

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots\right] + a_1 x$$

SNB Check: $(x^2+1)y'' + xy' - y = 0$, Series solution is:
$y(x) = y(0) + y'(0)x + \left(\frac{1}{2}y(0)\right)x^2 + \left(-\frac{1}{8}y(0)\right)x^4 + \left(\frac{1}{16}y(0)\right)x^6 + \left(-\frac{5}{128}y(0)\right)x^8 + O\left(x^9\right)$

Ma 227 Final Exam 95S
Problem 1
Find the first four nonzero terms of the Fourier *sine* series of

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ -2 & \pi < x < 2\pi \end{cases}$$

Solution:
If $f(x)$ is a function defined on $[0, L]$, then its Fourier sine expansion is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Here $L = 2\pi$ so that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{nx}{2}\right)$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\left(\frac{nx}{2}\right)dx$$

Hence

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} 0 \sin\frac{nx}{2}dx + \int_{\pi}^{2\pi}(-2)\sin\frac{nx}{2}dx\right]$$
$$= \frac{1}{\pi}(4)\left[\frac{\cos\pi n - \cos\left(\frac{1}{2}\pi n\right)}{n}\right]$$

.

Evaluating this last expression for $n = 1, 2, 3, 4, 5$ we get

$n = 1 \quad a_1 = \frac{4}{\pi}[-1]$

$n = 2 \quad a_2 = \frac{4}{\pi}[1]$

$n = 3 \quad a_3 = \frac{4}{\pi}\left[-\frac{1}{3}\right] = -\frac{4}{3\pi}$

$n = 4 \quad a_4 = 0$

$n = 5 \quad a_5 = \frac{4}{\pi}\left[-\frac{1}{5}\right] = -\frac{4}{5\pi}$

Thus $f(x) = -\frac{4}{\pi}\sin\frac{x}{2} + \frac{4}{\pi}\sin x - \frac{4}{\pi}\sin\frac{3x}{2} + 0\sin 2x - \frac{4}{\pi}\sin\frac{5x}{2} + \cdots$

b) (8 points)

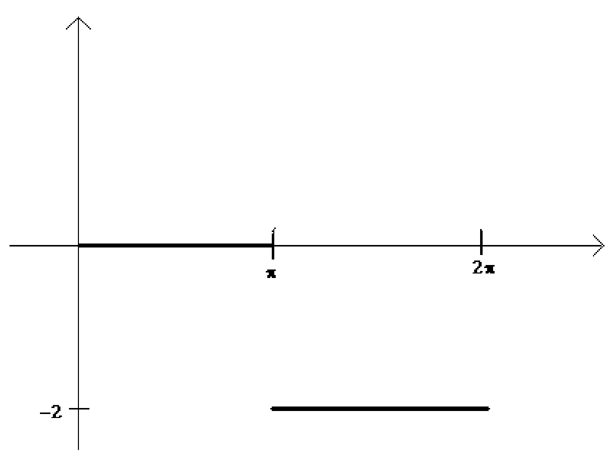
Sketch the graph of the function to which the Fourier sine series of the function

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ -2 & \pi < x < 2\pi \end{cases}$$

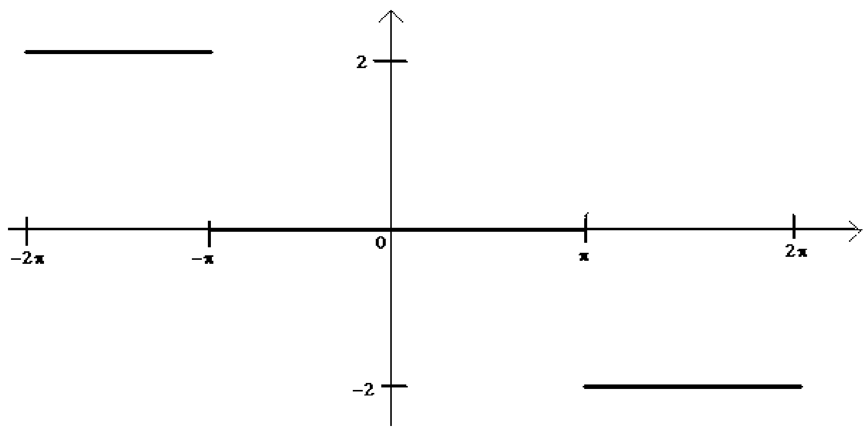
converges on $-2\pi < x < 4\pi$.

Solution

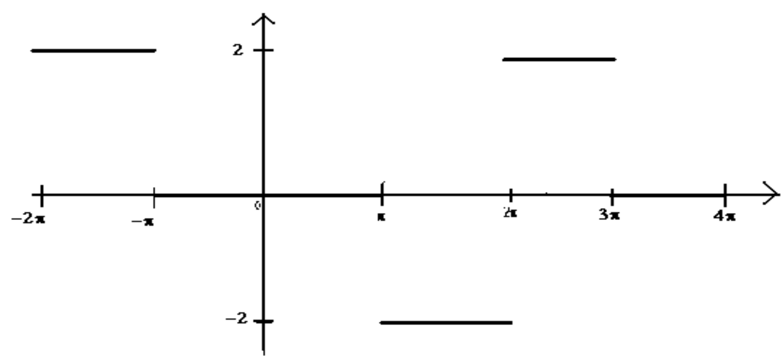
The graph of the given function is below.



Since we were asked to find the Fourier sine expansion of $f(x)$, this means that we are seeking an odd expansion of f. Hence the graph above is reflected first across the $y-axis$, and then across the $x-axis$ to get an odd function. The result is given below.



The Fourier sine series generates an odd function with period $2L$. Here $L = 2\pi$, so the function generated by the Fourier series has period $2(2\pi) = 4\pi$. Since the last graph above given the function on the interval $[-2\pi, 2\pi]$, i.e., on an interval of length 4π, we may move this graph either to the left or the right to get the function anywhere. Thus we have



c) (9 points)
Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0, \;\; y'(0) = y'(1) = 0$$

Be sure to check the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

Solution

I. Consider the case $\lambda < 0$ first. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. The DE becomes

$$y'' - \alpha^2 y = 0.$$

The general solution of this equation is

$$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

. Thus

$$y'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x}$$

$y'(0) = c_1\alpha - c_2\alpha = 0$ and $y'(1) = c_1 \alpha e^{\alpha} - c_2 \alpha e^{-\alpha} = 0.$

The first equation implies that $c_1 = c_2$. Thus the second equation becomes $c_1(e^{\alpha} - e^{-\alpha}) = 0.$ Thus $c_1 = 0$, this tells us that $c_2 = 0$ also. Therefore $y = 0$ is the only solution if $\lambda < 0$.

II. Suppose $\lambda = 0.$ The DE becomes $y'' = 0$ which has the solution $y = c_1 x + c_2.$ The boundary conditions imply $c_1 = 0,$ so that $y = c_2.$ Thus $y = c_2$ where $c_2 \neq 0$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$.

III. Suppose $\lambda > 0.$ Let $\lambda = \beta^2$ where $\beta \neq 0.$ The DE becomes

$$y'' + \beta^2 y = 0.$$

The general solution of this equation is

$$y(x) = c_1 \sin\beta x + c_2 \cos\beta x.$$

Thus

$$y'(x) = c_1 \beta \cos\beta x - c_2 \beta \sin\beta x$$

Now $y'(0) = c_1\beta = 0$ Since $\beta \neq 0,$ we must have $c_1 = 0.$ Thus $y(x) = c_2 \cos\beta x.$ Now $y'(x) = -c_2\beta\sin\beta x$ and $y'(1) = -c_2\beta\sin\beta = 0.$ For a nontrivial solution we must have $c_2 \neq 0.$ This means that $\sin\beta = 0$ or $\beta = n\pi,$ $n = 1, 2, 3, \ldots$ The eigenvalues are therefore $\lambda = \beta^2 = n^2\pi^2$ and the corresponding eigenfunctions are $y_n = a_n \cos n\pi x,$ $n = 1, 2, 3, \ldots$

We may also include the eigenfunction found in II above by allowing n to equal 0. Hence all of the eigenfunctions are given by $y_n = a_n \cos n\pi x,$ $n = 0, 1, 2, 3, \ldots$ with corresponding eigenvalues $\lambda = n^2\pi^2,$ $n = 0, 1, 2, 3, \ldots$

Problem 2

a) (10 points)

Use separation of variables, $u(x,t) = X(x)T(t)$, to find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is to be a solution of

$$11t^2x^9u_{xx} - (t-3)(x+2)u_{ttt} = 0$$

Solution:

$$u(x,t) = X(x)T(t)$$

$u_x = X'T, \qquad u_{xx} = X''T, \qquad u_t = XT'$, etc.

Thus the given equation becomes

$$11t^2x^9X''T - (t-3)(x+2)XT''' = 0$$

$\Rightarrow$

$$11x^9\frac{X''}{(x+2)X} = (t-3)\frac{T'''}{t^2T} = k, \qquad k \text{ a constant}$$

This yields the two ODEs

$$11x^9X'' - k(x+2)X = 0$$
$$(t-3)T''' - kt^2T = 0$$

b) (15 points)
Solve:

P.D.E.: $u_{xx} - 4u_{tt} = 0$

B.C.'s: $u_x(0,t) = 0 \qquad u_x(\pi,t) = 0$

I.C.'s: $u(x,0) = 0 \qquad u_t(x,0) = -8\cos(4x) + 17\cos(8x)$

Solution:

Let $u(x,t) = X(x)T(t)$. Then differentiating and substituting in the PDE yields

$X''T = 4XT''$

$\Rightarrow \qquad \dfrac{X''}{X} = 4\dfrac{T''}{T}$

Using the argument that the left hand side is purely a function of *x* and the right hand side is purely a function of *t*, and the only way that they can be equal is if they are equal to a constant, we get

$\dfrac{X''}{X} = 4\dfrac{T''}{T} = k \qquad k$ a constant

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T'' - \frac{1}{4}kT = 0$$

The boundary condition $u_x(0,t) = 0$ implies, since $u_x(x,t) = X'(x)T(t),$ that $X'(0)T(t) = 0.$ We cannot have $T(t) = 0,$ since this would imply that $u(x,t) = 0$. Thus $X'(0) = 0$. Similarly, the boundary condition $u_x(\pi,t) = 0$ leads to $X'(\pi) = 0$.

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X'(0) = X'(\pi) = 0$$

$k = -n^2 \qquad X_n(x) = a_n \cos nx \qquad n = 1,2,3,\ldots$
Note that for $k = 0$ we have $X'' = 0$ so $X(x) = c_1 x + c_2$. The conditions $X'(0) = X'(\pi) = 0$ imply that $c_1 = 0$ so for this case $X(x) = c_2$ any nonzero constant. We may combine the cases $k = 0$ and $k = -n^2$ by writing

$$X_n(x) = a_n \cos nx \quad n = 0,1,2,\ldots$$

Substituting $k = -n^2$ into the equation for $T(t)$ leads to

$$T'' + \frac{n^2}{4} T = 0$$

which has the solution $T_n(t) = b_n \sin \frac{nt}{2} + c_n \cos \frac{nt}{2},\ n = 1,2,3,\ldots$

The initial condition $u(x,0) = 0$ implies $X(x)T(0) = 0$ so that $T(0) = 0.$ Thus $c_n = 0$.

We now have the solutions

$$u_n(x,t) = A_n \cos nx \sin \frac{nt}{2} \qquad n = 1,2,3,\ldots$$

When $k = 0$ then $T(t) = b_0 t + d$. Since $T(0) = 0 \Rightarrow d = 0$ and $T_0(t) = b_0 t$.

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = A_0 t + \sum_{n=1}^{\infty} A_n \cos nx \sin \frac{nt}{2}$$

satisfies the PDE, the boundary conditions, and the first initial condition. Since

$$u_t(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{n}{2}\right) \cos nx \cos \frac{nt}{2}$$

The last initial condition leads to

$$u_t(x,0) = -8\cos(4x) + 17\cos(8x) = A_0 + \sum_{n=1}^{\infty} A_n\left(\frac{n}{2}\right)\cos nx.$$

Matching the cosine terms on both sides of this equation leads to

$A_4\left(\frac{4}{2}\right) = -8$ so that $A_4 = -4$ and $A_8\left(\frac{8}{2}\right) = 17$ so that $A_8 = \frac{17}{4}$. All of the other constants must be zero, since there are no cosine terms on the left to match with. Also, $A_0 = 0$ since there is no constant term in the given initial condition. Thus

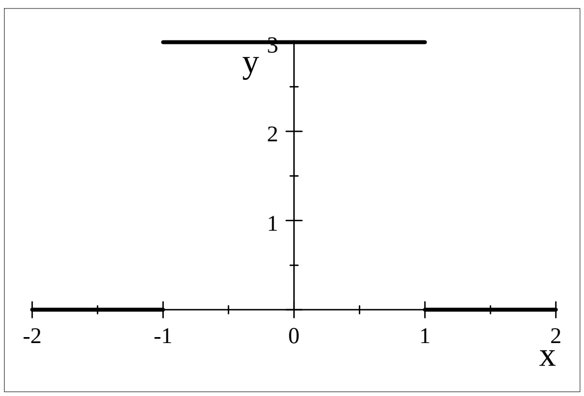
$$u(x,t) = -4\cos 4x\sin\frac{4t}{2} + \frac{17}{4}\cos 8x\sin\frac{8t}{2} = -4\cos 4x\sin 2t + \frac{17}{4}\cos 8x\sin 4t$$

Ma 227 Final Exam 97F

1. a) Find the first four non-zero terms on the Fourier cosine series of

$$f(x) = \begin{cases} 3 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

3



Cosine Formula:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad n = 0,1,2,\ldots$$

Note: Prof Levine gave the Fourier cosine formulas as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \quad n = 1, 2, \ldots$$

These two sets of formulas are the same.

$$a_0 = 1\left[\int_0^1 3dx + \int_1^2 0dx \right] = 3$$

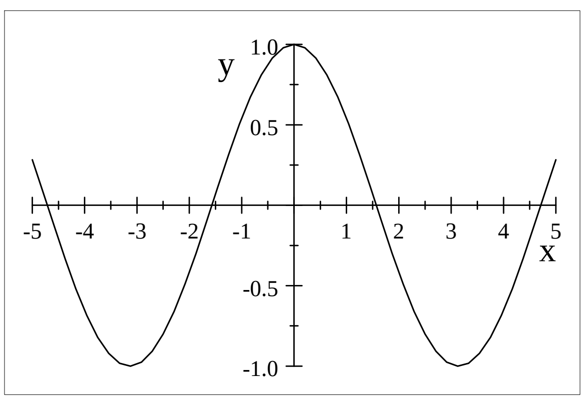
$$a_n = \int_0^1 3\cos \frac{n\pi x}{2} dx + \int_1^2 0\cos \frac{n\pi x}{2} dx = \frac{6}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^1 = \frac{6}{n\pi} \sin \frac{n\pi}{2}$$

$$a_n = \begin{cases} \frac{6}{n\pi}(-1)^{\frac{n-1}{2}} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

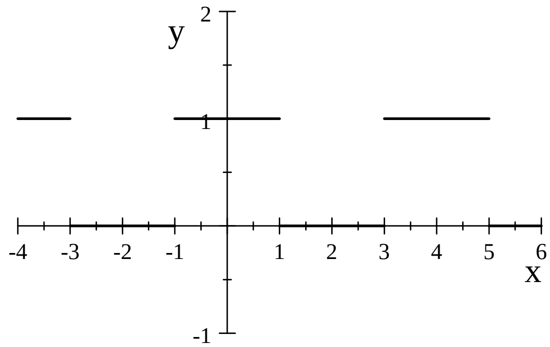
Thus $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2} = \sum_{n=1}^{9} \frac{6(-1)^{n+1}}{(2n-1)\pi} \cos \frac{(2n-1)\pi x}{2}$

computing the first few terms: $f(x) = \frac{3}{2} + \frac{6}{\pi} \cos \frac{1}{2}\pi x - \frac{2}{\pi} \cos \frac{3}{2}\pi x + \frac{6}{5\pi} \cos \frac{5}{2}\pi x - \frac{6}{7\pi} \cos \frac{7}{2}\pi x + \frac{2}{3\pi} \cos \frac{9}{2}\pi x$

$\cos x$



1. b) Sketch the graph of $f(x)$ on $-4 < x < 6$



1. c) Solve the boundary value problem:

$$y''(x) - y(x) = x; \qquad y(0) = 0; \qquad y'(1) = 1$$

homogeneous solution: $y''(x) - y(x) = 0$

characteristic equation: $r^2 - 1 = 0 \Rightarrow r = \pm 1$

$$y(x) = c_1 e^x + c_2 e^{-x}$$

particular solution: $\left.\begin{array}{l} y(x) = Ax + B \\ y'(x) = A \\ y''(x) = 0 \end{array}\right\} \Rightarrow A = -1 \Rightarrow y(x) = -x$

general solution: $y(x) = c_1 e^x + c_2 e^{-x} - x$ then $y'(x) = c_1 e^x - c_2 e^{-x} - 1$

B.C. $\Rightarrow$ $y(0) = c_1 e^0 + c_2 e^{-0} - 0 = 0 \Rightarrow c_1 = -c_2$

and $y'(1) = c_1 e^1 - c_2 e^{-1} - 1 = 1 \Rightarrow c_1 e - c_2 e^{-1} = 2$

$\begin{array}{l} c_1 = -c_2 \\ c_1 e - c_2 e^{-1} = 2 \end{array}$, Solution is : $\left\{ c_2 = -\frac{2}{e+e^{-1}}, c_1 = \frac{2}{e+e^{-1}} \right\}$,

So

$$y(x) = \frac{2}{e+e^{-1}} e^x - \frac{2}{e+e^{-1}} e^{-x} - x$$

2. a) Use separation of variables, $u(r,\theta) = R(r)T(\theta)$, to find ordinary differential equations which $R(r)$ and $T(\theta)$ must satisfy if $u(r,\theta)$ is to be a solution of

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

Do *not* solve these equations.

Solution: Let $u(r,\theta) = R(r)T(\theta)$ then $\quad u_r = R'(r)T(\theta) \quad u_{rr} = R''(r)T(\theta) \quad u_{\theta\theta} = R(r)T''(\theta)$

and $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ becomes

$$R''(r)T(\theta) + \frac{1}{r} R'(r)T(\theta) + \frac{1}{r^2} R(r)T''(\theta) = 0$$

$$r^2 R''(r)T(\theta) + rR'(r)T(\theta) = -R(r)T''(\theta)$$

$$\frac{r^2R''(r) + rR'(r)}{-R(r)} = \frac{T''(\theta)}{T(\theta)} = k$$

since R and T are independent
resulting in the equations

$$r^2R''(r) + rR'(r) + kR(r) = 0$$

and

$$T''(\theta) - kT(\theta) = 0$$

2. b)Consider the non-homogeneous problem

P.D.E.: $u_{xx} = 9u_t$

B.C.'s: $u_x(0,t) = 0 \qquad u(1,t) = 2$

I.C.: $u(x,0) = -3\cos\frac{7\pi}{2}x + 2$

i) (5 points)

Let $v(x,t) = u(x,t) - 2$ and show that $v(x,t)$ satisfies the homogeneous problem

P.D.E.: $v_{xx} = 9v_t$

B.C.: $v_x(0,t) = 0 \qquad v(1,t) = 0$

I.C.: $v(x,0) = -3\cos\frac{7\pi}{2}x$

Solution to i) Since $u(x,t) = v(x,t) + 2$

$u_x(x,t) = v_x(x,t) \qquad u_{xx}(x,t) = v_{xx}(x,t) \qquad u_t(x,t) = v_t(x,t)$

$u(1,t) = 2$ and $u(x,t) = 2 = v(x,t) + 2 \Rightarrow v(1,t) = 0$

$u_x(0,t) = 0 \Rightarrow v_x(0,t) = 0$

$u(x,0) = -3\cos\frac{7\pi}{2} + 2 = v(x,0) + 2 \Rightarrow v(x,0) = -3\cos\frac{7\pi}{2}$

2. b) ii) (10 points)

Solve the above problem for $v(x,t)$.

Solution to ii) Let $v(x,t) = X(x)T(t)$

then $X''T = 9XT' \Rightarrow \frac{X''}{X} = 9\frac{T'}{T} = k$

resulting in the ordinary differential equations:

$X'' - kX = 0$ and $T' - \frac{k}{9}T = 0$

Boundary Conditions become: $X'(0)T(t) = 0$ and $X(1)T(t) = 0$

$\Rightarrow X'(0) = 0$ and $X(1) = 0$

Solving the differential equation $X'' - kX = 0$ consider all values of k

$k < 0$ let $k = -u^2; \quad u > 0$

$X'' + u^2X = 0$ has the solution: $X(x) = c_1\cos ux + c_2\sin ux$

and $X'(x) = -c_1u\sin ux + c_2u\cos ux$

B.C. $\Rightarrow X(1) = c_1\cos u + c_2\sin u = 0$ and $X'(0) = c_2u = 0$

$\Rightarrow c_2 = 0$ thus $c_1\cos u = 0 \Rightarrow u_n = \frac{(2n-1)\pi}{2} \quad n = 1,2,\ldots$

$$\Rightarrow k_n = -\frac{(2n-1)^2\pi^2}{4} \quad n = 1,2,\ldots$$

$$\text{so} \quad X_n(x) = c_n \cos\frac{(2n-1)\pi}{2}x$$

$k = 0 \Rightarrow X'' = 0$ which has the solution: $X(x) = c_1 x + c_2$ and $X'(x) = c_1$

B.C. $\Rightarrow X(1) = c_1 + c_2 = 0$ and $X'(0) = c_1 = 0 \Rightarrow c_2 = 0$

thus $X(x) \equiv 0$ is the trivial solution.

$k > 0$ let $k = u^2$; $u > 0$

$X'' - u^2 X = 0$ has the solution: $X(x) = c_1 e^{ux} + c_2 e^{-ux}$

and $X'(x) = c_1 u e^{ux} - c_2 u e^{-ux}$

B.C. $\Rightarrow X'(0) = c_1 u - c_2 u = 0 \Rightarrow c_1 = c_2$

and $X(1) = c_1 e^u + c_2 e^{-u} = 0 \Rightarrow c_1 e^u + c_1 e^{-u} = 0 \Rightarrow c_1(e^u + e^{-u}) = 0$

$\Rightarrow c_1 = c_2 = 0$ thus $X(x) \equiv 0$ is the trivial solution.

Using the non-trivial solution $k_n = -\frac{(2n-1)^2\pi^2}{4}$ $\quad X_n(x) = c_n \cos\frac{(2n-1)\pi}{2}x,$

the equation $T' - \frac{k}{9}T = 0$ becomes $T' + \frac{(2n-1)^2\pi^2}{36}T = 0$

solving by separating $\frac{T'}{T} = -\frac{(2n-1)^2\pi^2}{36} \Rightarrow \int \frac{T'}{T} = -\int \frac{(2n-1)^2\pi^2}{36}$

$\Rightarrow \ln T = -\frac{(2n-1)^2\pi^2}{36}t + c \Rightarrow T_n(t) = c_n e^{-\frac{(2n-1)^2\pi^2}{36}t}$

Therefore $v_n(x,t) = X_n(x)T_n(t)$

$v_n(x,t) = c_n \cos\frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36}t}$

so

$$v(x,t) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{36}t}$$

Using I.C. to compute coefficients:

$$v(x,0) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2} = -3\cos\frac{7\pi x}{2}$$

by equating coefficients: $c_1 = 0, c_2 = 0, c_3 = -3, c_4 = 0, \ldots$

$$v(x,t) = -3\cos\frac{7\pi x}{2} e^{-\frac{49\pi^2}{36}t}$$

is the solution.

2. b)iii) (2 points)

Now use the results of b) i) and ii) to find $u(x,t)$.

Solution to iii)

$$u(x,t) = -3\cos\frac{7\pi x}{2} e^{-\frac{49\pi^2}{36}t} + 2$$

Ma 227 Final Exam 98S

Problem 1

a) (8 points)
Find the first four nonzero terms of the Fourier *cosine* series of

$$f(x) = \begin{cases} -1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

Solution
If $f(x)$ is a function defined on $[0, L]$, then its Fourier cosine expansion is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad n = 1, 2, 3, \ldots$
Here $L = \pi$ so that $f(x) = \sum_{n=1}^{\infty} a_n \cos(nx), a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$.

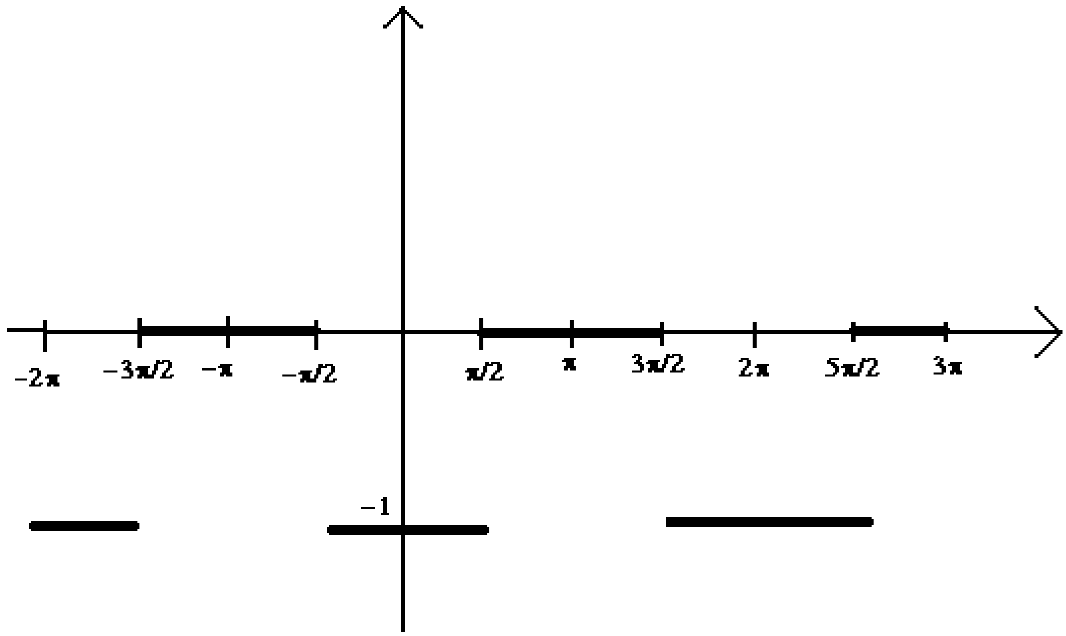
Thus $a_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(-1)dx + \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(0)dx = -\frac{1}{2}$. Also,

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}(-1)\cos nxdx = -\frac{2}{n\pi}[\sin nx]_0^{\frac{\pi}{2}} = -\frac{2}{n\pi}\left[\sin\frac{n\pi}{2}\right]$$

Therefore
$a_1 = -\frac{2}{\pi}, \quad a_2 = 0, \quad a_3 = +\frac{2}{3\pi}, \quad a_4 = 0, \quad a_5 = -\frac{2}{5\pi}, \quad a_6 = 0, \quad a_7 = +\frac{2}{7\pi}$

Hence
$f(x) = -\frac{1}{2} - -\frac{2}{\pi}\cos x + 0 \bullet \cos 2x + \frac{2}{3\pi}\cos 3x + 0 \bullet \cos 4x - \frac{2}{5\pi}\cos 5x + 0 \bullet \cos 6x + \frac{2}{7\pi}\cos 7x + \cdots$

b) (8 points)
Sketch the graph of the function to which the Fourier series in (a) converges
on $-2\pi < x < 3\pi$.



c) (9 points)
Find the eigenvalues and eigenfunctions for the problem

$y'' + \lambda y = 0$; $y(0) = 0$; $y(2) = 0$

Be sure to check the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.
I. Consider the case $\lambda < 0$ first. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. The DE becomes

$y'' - \alpha^2 y = 0.$

The general solution of this equation is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Thus

$y(0) = c_1 + c_2 = 0$ and $y(2) = c_1 e^{2\alpha} + c_2 e^{-2\alpha} = 0.$

The first equation implies that $c_1 = -c_2$. Thus the second equation becomes $c_1\left(e^{2\alpha} + e^{-2\alpha}\right) = 0$. Thus $c_1 = 0$; this tells us that $c_2 = 0$ also. Therefore $y = 0$ is the only solution if $\lambda < 0$. Hence there are no negative eigenvalues.

II. Suppose $\lambda = 0$. The DE becomes $y'' = 0$ which has the solution $y = c_1 x + c_2$. The boundary conditions imply $y(0) = c_1 = 0$, so that $y = c_2$. But $y(2) = c_2 = 0$ so that $y = 0$. Hence there is no eigenfunction corresponding to the eigenvalue $\lambda = 0$.

III. Suppose $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$y'' + \beta^2 y = 0.$

The general solution of this equation is $y(x) = c_1 \sin \beta x + c_2 \cos \beta x.$ Thus

Now $y(0) = c_2 = 0$ Thus $y(x) = c_2 \sin \beta x.$ Now $y(2) = c_2 \sin 2\beta = 0.$ For a nontrivial solution we must have $c_2 \neq 0.$ This means that $\sin 2\beta = 0$ or $\beta = \frac{n\pi}{2},\ n = 1,2,3,\ldots$ The eigenvalues are therefore $\lambda = \beta^2 = \frac{n^2\pi^2}{4}$ and the corresponding eigenfunctions are $y_n = a_n \sin \frac{n\pi}{2}x,\ n = 1,2,3,\ldots$

Problem 2

a) (10 points)

Use separation of variables, $u(x,t) = X(x)T(t)$, to find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is to be a solution of

$$5x^5t^2u_{tt} + (t+3)^5(x+5)^2u_{xx} = 0$$

Do *not* solve these equations.

Solution:

$u_x = X'T, \qquad u_{xx} = X''T, \qquad u_t = XT', \ u_{tt} = XT''$

Thus the given equation becomes

$15t^2x^5XT'' + (t+3)^5(x+5)^2X''T = 0$

$$\Rightarrow \qquad 15x^5\frac{X}{(x+5)^2X''} = -(t+3)^5\frac{T}{t^2T''} = k, \qquad k \text{ a constant}$$

This yields the two ODEs

$15x^5X - k(x+5)^2X'' = 0$

$(t+3)^5T + kt^2T'' = 0$

b) (15 points)

Solve:

$$\begin{aligned} \text{P.D.E.: } & u_{xx} = 4u_t \\ \text{B.C.s: } & u(0,t) = u(2,t) = 0 \\ \text{I.C.: } & u(x,0) = -3\sin\frac{\pi x}{2} + 23\sin\pi x - 4\sin 2\pi x \end{aligned}$$

Let $u(x,t) = X(x)T(t).$ Then differentiating and substituting in the PDE yields

$$X''T = 4XT'$$
$$\Rightarrow \frac{X''}{X} = 4\frac{T'}{T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T'}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T' - \frac{1}{4}kT = 0$$

The boundary condition $u(0,t) = 0$ implies that $X(0)T(t) = 0.$ We cannot have $T(t) = 0,$ since this would imply that $u(x,t) = 0$. Thus $X(0) = 0.$ Similarly, the boundary condition $u(2,t) = 0$ leads to $X(2) = 0.$

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X(0) = X(2) = 0$$

This boundary value problem is the one given in Problem 1(c) above with $k = -\lambda$. The solution is

$$k = -\left(\frac{n\pi}{2}\right)^2 \qquad X_n(x) = a_n \sin\frac{n\pi}{2}x \qquad n = 1,2,3,\ldots$$

Substituting the values of k into the equation for $T(t)$ leads to

$$T' + \frac{n^2\pi^2}{16}T = 0$$

which has the solution $T_n(t) = c_n e^{-\frac{n^2\pi^2 t}{16}}$, $n = 1,2,3,\ldots$

We now have the solutions

$$u_n(x,t) = A_n \sin\frac{n\pi}{2}x e^{-\frac{n^2\pi^2 t}{16}} \qquad n = 1,2,3,\ldots$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{2}x e^{-\frac{n^2\pi^2 t}{16}}$$

satisfies the PDE and the boundary conditions. Since

$$u(x,0) = -3\sin\frac{\pi x}{2} + 23\sin\pi x - 4\sin 2\pi x = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{2}x.$$

Matching the cosine terms on both sides of this equation leads to

$A_1 = -3$ $A_2 = 23$ and $A_4 = -4.$ All of the other constants must be zero, since there are no sine terms on the left to match with them. Thus

$$u(x,t) = -3\sin\frac{\pi x}{2}e^{-\frac{\pi^2}{16}t} + 23\sin\pi x e^{-\frac{\pi^2}{4}t} - 4\sin 2\pi x e^{-\pi^2 t}$$