Review for Ma 221 Final Exam

The Ma 221 Final Exam from December 1995

Solutions to selected problems

1.a) Solve the initial value problem
$$(2x\cos y + 3x^2 y)dx + (x^3 - x^2\sin y - y)dy = 0$$
$$y(0) = 2$$

The equation is first order, for which we have techniques for four types, viz. separable, linear, exact and Bernoulli. It is not separable, linear or a Bernoulli d.e., so we will test to see if it's exact. P(x,y)dx + Q(x,y)dy is exact if there is a function $F(x,y)$ such that
$$dF = P(x,y)dx + Q(x,y)dy$$
The test is that when this is true, then
$$P_y = Q_x.$$

Here $P = (2x\cos y + 3x^2 y)$ and $Q = (x^3 - x^2\sin y - y)$. Calculating the partial derivatives, we have $P_y = -2x\sin y + 3x^2$ and $Q_x = 3x^2 - 2x\sin y$ which are equal.

So the equation is exact and we proceed to find F. Then implicit solutions of the d.e. will be given by $F(x,y) = c$. We must have that $F_x = P$ and $F_y = Q$.

So $\frac{\partial F}{\partial x} = P = (2x\cos y + 3x^2 y)$. Hence $F = x^2\cos y + x^3 y + g(y)$, where g(y) is any function of y. Now taking the partial derivative of F with respect to y,

$\frac{\partial F}{\partial y} = -2x\sin y + x^3 + g'(y) = Q = (x^3 - x^2\sin y - y)$. Thus, we have that $g'(y) = -y$ and such a $g(y)$ is $g(y) = -(\frac{1}{2})y^2$, and the solution of the d.e. is given by

$F(x,y) = x^2\cos y + x^3 y - (\frac{1}{2})y^2 = c$. Finally we will use the initial condition to find c. Substituting into the equation, we have

$F(0,2) = 0 * \cos 2 + 0 * 2 - (\frac{1}{2})2^2 = -2 = c.$

In conclusion, the solution is given implicitly by $\boxed{x^2\cos y + x^3 y - (\frac{1}{2})y^2 = -2}$.

1.b) Find the general solution to $4xy + (x^2 + 1)y' = x$.

There are two apparent ways to solve this problem. It is both separable and linear.

Separable Approach: First we write the d.e. in the form $(x^2 + 1)y' = x - 4xy = x(1 - 4y)$.

Thus $\frac{1}{1-4y}\frac{dy}{dx} = \frac{x}{(x^2+1)}$. Integration yields $(\frac{-1}{4})\ln|1-4y| = (\frac{1}{2})\ln(x^2+1)+c$. I.e., $\ln|1-4y| = \ln(x^2+1)^{-2} - 4c$.

Thence, $|1-4y| = e^{\ln(x^2+1)^{-2}-4c} = e^{-4c}e^{\ln(x^2+1)^{-2}} = e^{-4c}(x^2+1)^{-2}$. This could be written $4y-1 = \pm e^{-4c}(x^2+1)^{-2}$.

Finally, we have $\boxed{y = \frac{1}{4} + k(x^2+1)^{-2}}$, sweeping all of the constant fussing into a new arbitrary constant, k, and observing that this also yields a solution when k=0..

Linear Approach: For this the d.e. is put into the form $y' + P(x)y = Q(x)$, and the integrating factor is calculated.

The d.e. is $y' + (\frac{4x}{x^2+1})y = (\frac{x}{x^2+1})$. The integrating factor is $e^{\int P(x)dx} = e^{\int(\frac{4x}{x^2+1})dx} = e^{2\ln(x^2+1)} = e^{\ln(x^2+1)^2} = (x^2+1)^2$.

Multiplying the d.e. by the integrating factor produces $(x^2+1)^2y' + 4x(x^2+1)y = x(x^2+1)$.

This is equivalent to $\frac{d}{dx}((x^2+1)^2y) = x^3+x$. Integration produces $(x^2+1)^2y = \frac{1}{4}x^4 + \frac{1}{2}x^2 + c$.

Finally, we have $\boxed{y = \frac{1}{4}(x^4+2x^2)(x^2+1)^{-2} + c(x^2+1)^{-2}}$.

You can put this in the form obtained in the separable approach by setting $c = k + \frac{1}{4}$ and fussing a little more.

1.c) Find the general solution of $y' - \frac{3}{2x}y = \frac{2x}{y}$.

First we observe that this is a Bernoulli d.e. Multiplying both sides by y we get $yy' - \frac{3}{2x}y^2 = 2x$.

Let $v = y^2$. Then $v' = 2yy'$. Substituting into the d.e., we have $\frac{1}{2}v' - \frac{3}{2x}v = 2x$, which is linear.

In standard form, this is $v' - \frac{3}{x}v = 4x$. The integrating factor is $e^{\int(\frac{-3}{x})dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$.

Multiplying by the integrating factor yields $x^{-3}v' - 3x^{-4}v = 4x^{-2}$.

This can be rewritten as $\frac{d}{dx}(x^{-3}v) = 4x^{-2}$. Integrating we obtain $x^{-3}v = -4x^{-1} + c$.

I.e. $v = cx^3 - 4x^2$. Hence $y^2 = cx^3 - 4x^2$ or $\boxed{y = \pm\sqrt{cx^3 - 4x^2}}$.

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2. Find the general solution to the differential equations.

a) $y'' + y = 4x + 10\sin x$.

The auxiliary equation is $r^2 + 1 = 0$. The roots are $r = \pm i$.

So a general solution for the homogeneous equation is $y_h = c_1 \cos x + c_2 \sin x$.

To find a particular solution for $y'' + y = 4x$, we try $y_{p1} = ax + b$.

Since $y''_{p1} = 0$, substitution yields $ax + b = 4x$, hence $a = 4$ and $b = 0$.

For $y'' + y = 10\sin x$, we observe that the right hand is a solution of the homogeneous equation and thus we must seek a solution in the form
$y_{p2} = Ax\cos x + Bx\sin x$.
Then, $y'_{p2} = A\cos x - Ax\sin x + B\sin x + Bx\cos x$
and $y''_{p2} = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x$.
Substituting these into the d.e., we obtain
$-2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x + Ax\cos x + Bx\sin x = 10\sin x$
Equating the coefficients of $\sin x$ and $\cos x$, we obtain

$$\begin{matrix} -2A = 10 \\ 2B = 0 \end{matrix} \quad \text{or} \quad \begin{matrix} A = -5 \\ B = 0 \end{matrix}$$

Putting everything, our general solution is

$$\boxed{y = y_h + y_{p1} + y_{p2} = c_1 \cos x + c_2 \sin x + 4x - 5x\cos x}.$$

b) $y'' + 9y = \frac{1}{4}\csc 3x$

The auxiliary equation is $r^2 + 9 = 0$, whose roots are $r = \pm 3i$.

A general solution of the homogeneous equation is $y_h = c_1 \cos 3x + c_2 \sin 3x$.

For the non-homogeneous equation, we will use the method of variation of parameters.

I.e., we seek a solution of the form $y = v_1(x)\cos 3x + v_2(x)\sin 3x$.

Then $y' = v'_1 \cos 3x + v'_2 \sin 3x - 3v_1 \sin 3x + 3v_2 \cos 3x$.

We will assume that $\boxed{v'_1 \cos 3x + v'_2 \sin 3x = 0}$

Then $y'' = -3v_1' \sin 3x + 3v_2' \cos 3x - 9v_1 \cos 3x - 9v_2 \sin 3x.$

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Substituting into the d.e. produces

$$-3v_1' \sin 3x + 3v_2' \cos 3x - 9v_1 \cos 3x - 9v_2 \sin 3x + 9(v_1(x) \cos 3x + v_2(x) \sin 3x) = \frac{1}{4} \csc 3x$$

Simplifying and repeating the assumption, we have 2 equations for v_1 and v_2.

$$-3v_1' \sin 3x + 3v_2' \cos 3x = \frac{1}{4} \csc 3x$$

$$v_1' \cos 3x + v_2' \sin 3x = 0$$

Multiply the first equation by -$\sin 3x$, the second equation by $3 \cos 3x$ and add to obtain

$3v_1'(\sin^2 3x + \cos^2 3x) = \frac{-1}{4} \csc 3x \sin 3x$. I.e. $3v_1' = \frac{-1}{4}$.

So, $v_1 = \frac{-1}{12} x$. Substituting $v_1' = \frac{-1}{12}$ into the second equation yields

$\frac{-1}{12} \cos 3x + v_2' \sin 3x = 0$. I.e. $v_2' = \frac{\cos 3x}{12 \sin 3x}$.

So, $v_2 = \int \frac{\cos 3x}{12 \sin 3x} dx = \frac{1}{36} \ln(\sin 3x)$.

At last, we have a general solution in the form

$$y = (c_1 - \frac{1}{12} x) \cos 3x + (c_2 + \frac{1}{36} \ln(\sin 3x)) \sin 3x.$$

3.a) Find $\mathcal{L}^{-1}\left\{ \frac{3s}{(s-4)(s-2)} \right\}$

First, we must find the partial fractions expansion. Then, we can use the table to find the inverse Laplace transform.

$\frac{3s}{(s-4)(s-2)} = \frac{A}{s-4} + \frac{B}{s-2}$ Cross multiplying to clear the denominators yields

$3s = A(s-2) + B(s-4)$.

Let $s = 2$. Thus, we have $6 = B(-2)$ or $B = -3$.

Similarly with $s = 4$, we obtain $12 = A(2)$ or $A = 6$.

Now, from the table we obtain the final result that

$$\boxed{\mathcal{L}^{-1}\left(\frac{3s}{(s-4)(s-2)}\right) = -3e^{2t} + 6e^{4t}}$$

b) Use Laplace Transforms to solve the initial value problem

$$y'' - 2y' - 8y = 0$$
$$y(0) = 3$$
$$y'(0) = 6$$

Taking the Laplace Transform of the d.e., we obtain

$$s^2\hat{y} - sy(0) - y'(0) - 2[s\hat{y} - y(0)] - 8[\hat{y}] = 0$$

$$(s^2 - 2s - 8)\hat{y} - 3s - 6 + 6 = 0$$

$$(s^2 - 2s - 8)\hat{y} = 3s$$

$$\hat{y} = \frac{3s}{(s^2-2s-8)} = \frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

As before, we cross multiply and set to -2 and 4 to obtain the coefficients.

$$3s = A(s+2) + B(s-4)$$

$(s = -2)$ $-6 = -6B, B = 1$
$(s = 4)$ $12 = 6A, A = 2$

Hence, $\boxed{y = 2e^{4t} + e^{-2t}}$.

4.a) Find $\mathcal{L}^{-1}\left(\frac{s-2}{s^2+4s+8}\right)$

$$\mathcal{L}^{-1}\left(\frac{s-2}{s^2 4s+8}\right) = \mathcal{L}^{-1}\left(\frac{s-2}{s^2+4s+4+4}\right) = \mathcal{L}^{-1}\left(\frac{s-2}{(s+2)^2+2^2}\right) = \mathcal{L}^{-1}\left(\frac{s+2-4}{(s+2)^2+2^2}\right)$$
$$= \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+2^2}\right) - 2\mathcal{L}^{-1}\left(\frac{2}{(s+2)^2+2^2}\right) = e^{-2t}\cos 2t - 2e^{-2t}\sin 2t$$

b) Not applicable for this course.

5.a) Solve $\frac{dy}{dx} = 7x^6 y - y \qquad y(0) = 1$

The equation is both separable and linear.

Let's try separable first.

We write the d.e. in the form $\frac{1}{y}dy = (7x^6 - 1)dx$

Integrating, we have $\int \frac{1}{y}dy = \int(7x^6 - 1)dx$

$$\ln|y| = x^7 - x + c$$

$$y = \pm e^c e^{x^7 - x}$$

Applying the initial condition, we obtain $1 = \pm e^c$

Thus, the solution is $\boxed{y = e^{x^7 - x}}$

Alternatively, we can solve the d.e. as a linear d.e.

First we rewrite the equation in the form $\frac{dy}{dx} - (7x^6 - 1)y = 0$

So, the integrating factor is $IF = e^{-\int(7x^6 - 1)dx} = e^{-(x^7 - x)}$

Multiplying through by the integrating factor, gathering terms and integrating, we obtain

$$e^{-(x^7 - x)}\frac{dy}{dx} - (7x^6 - 1)e^{-(x^7 - x)}y = 0$$

$$\frac{d}{dx}(e^{-(x^7 - x)}y) = 0$$

$$e^{-(x^7 - x)}y = c$$

$$y = ce^{(x^7 - x)}$$

Finally, we apply the initial condition to find $y(0) = 1 = c$.

As before, we have the solution in the form $\boxed{y = e^{-(x^7 - x)}}$

b) Find the general solution to $y'' - 3y' + 2y = 4e^{3x} + e^{2x}$.
The auxiliary equation is $r^2 - 3r + 2 = (r - 2)(r - 1) = 0$. So the roots are $r = 1, 2$.

Therefore, a general solution to the homogeneous equation is $y_h = c_1e^x + c_2e^{2x}$.

Since one of the terms on the right hand side, e^{2x}, is a solution of the homogeneous equation, we seek a particular solution in the form

$$y_p = Ae^{3x} + Bxe^{2x}$$
$$y_p' = 3Ae^{3x} + B(2xe^{2x} + e^{2x})$$
$$y_p'' = 9Ae^{3x} + B(4xe^{2x} + 2e^{2x} + 2e^{2x})$$

Substituting into the d.e. and gathering like terms produces

$$9Ae^{3x} + B(4xe^{2x} + 4e^{2x}) - 3[3Ae^{3x} + B(2xe^{2x} + e^{2x})] + 2[Ae^{3x} + Bxe^{2x}] = 4e^{3x} + e^{2x}$$

$$[9 - 9 + 2]Ae^{3x} + [4x + 4 - 6x - 3 + 2x]Be^{2x} = 4e^{3x} + e^{2x}$$

$$2Ae^{3x} + Be^{2x} = 4e^{3x} + e^{2x} \quad \text{I.e.} \quad A = 2, B = 1$$

Combining everything, we have

$$\boxed{y = y_h + y_p = c_1e^x + c_2e^{2x} + 2e^{3x} + xe^{2x}}.$$

6. a) Not applicable for Fall 2002 course.

b) Solution to be provided by Prof. Levine.

7.a) Solve $(\cos y)y' - 2x\sin y = -2x$

The hint is to let $z = \sin y$. Then $z' = (\cos y)y'$.

Substituting, we have $z' - 2xz = -2x$.

Once again, we have a first order d.e. which is both separable and linear.

The separable approach is as follows:

$$z' = 2xz - 2x = 2x(z - 1)$$

$$\frac{1}{z-1}dz = 2xdx$$

$$\int \frac{1}{z-1}dz = \int 2xdx$$

$$\ln|z - 1| = x^2 + c$$

$z - 1 = \pm e^c e^{x^2} = ke^{x^2}$, noting that k=0 also gives a solution.

We will undo the substitution after trying the other (linear) approach.

Here, we start with the d.e. in the form $z' - 2xz = -2x$.

The integrating factor is $IF = e^{\int -2x dx} = e^{-x^2}$

Multiplication by the integrating factor and integrating yields

$$e^{-x^2} z' - 2xe^{-x^2} z = -2xe^{-x^2}$$

$$\frac{d}{dx}(e^{-x^2} z) = -2xe^{-x^2}$$

$$.\int \frac{d}{dx}(e^{-x^2} z) dx = -\int 2xe^{-x^2} dx$$

$$e^{-x^2} z = e^{-x^2} + c$$

$$z = 1 + ce^{x^2}$$

So we have the same result and conclude with the steps

$$z = \sin y = 1 + ce^{x^2}$$

$$\boxed{y = \arcsin(1 + ce^{x^2})}$$

b) Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+3)}\right\}$

First, we must find the partial fractions decomposition, as follows:

$$\frac{1}{s^2(s^2+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+3}$$

$$1 = As(s^2 + 3) + B(s^2 + 3) + (Cs + D)s^2$$

Setting $s = 0$ yields $1 = 3B$, i.e. $B = \frac{1}{3}$.

Next, equating the coefficients of the various powers of s gives the following:

$s \qquad 0 = 3A$

$s^2 \qquad 0 = B + D$

$s^3 \qquad 0 = A + C$

So, $A = 0, \qquad C = -A = 0, \qquad D = -B = -\frac{1}{3}$

$$\frac{1}{s^2(s^2+3)} = \frac{1}{3s^2} - \frac{1}{3(s^2+3)}$$

$$\boxed{\mathcal{L}\left\{\frac{1}{s^2(s^2+3)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\} = \frac{1}{3}\left[t - \frac{1}{\sqrt{3}}\sin(\sqrt{3}\,t)\right]}$$

8.a) Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

Since $\frac{1}{2}t^2 = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$, we have $\frac{1}{s^3} = \int_0^\infty \frac{1}{2}t^2 e^{-st}dt = \int_0^\infty \frac{1}{2}u^2 e^{-su}du$

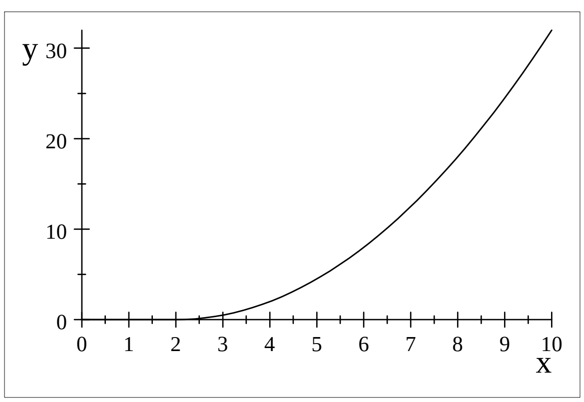
Then, $\frac{e^{-2s}}{s^3} = e^{-2s}\int_0^\infty \frac{1}{2}u^2 e^{-su}du = \int_0^\infty \frac{1}{2}u^2 e^{-s(u+2)}du$

In this integral, let $t = u + 2$.

Thus, we have $\frac{e^{-2s}}{s^3} = \int_2^\infty \frac{1}{2}(t-2)^2 e^{-st}dt$

So, we can interpret this as the Laplace transform of the function

$$\boxed{f(t) = \begin{cases} \frac{1}{2}(t-2)^2 & if \quad t > 2 \\ 0 & if \quad t < 2 \end{cases}}$$



b) Show that $\frac{1}{x^2}$ is an integrating factor for the differential equation $(2x^2 + y)dx + (x^2y - x)dy = 0$

and solve the differential equation.

By an integrating factor, we mean a function such that when we multiply the d.e. by the function, the equation is more easily integrated, i.e. solved. Let's multiply through by the given function.

$$(2+\frac{y}{x^2})dx+(y-\frac{1}{x})dy = 0$$

This is neither linear nor separable, so we test it for exactness.

$\frac{\partial}{\partial y}(2+\frac{y}{x^2}) = \frac{1}{x^2}$ $\frac{\partial}{\partial x}(y-\frac{1}{x}) = \frac{1}{x^2}$. These are equal, so we have an exact d.e.

We seek F(x,y) such that dF is the given expression.

$\frac{\partial F}{\partial x} = 2+\frac{y}{x^2}$ yields $F = 2x-\frac{y}{x}+g(y)$

Then $\frac{\partial F}{\partial y} = -\frac{1}{x}+g'(y) = y-\frac{1}{x}$, so $g'(y) = y$ and $g(y) = \frac{1}{2}y^2$

Finally, the solution is given implicitly by $\boxed{F(x,y) = 2x-\frac{y}{x}+\frac{1}{2}y^2 = c.}$

9.a) Not applicable for our course.

b) Find the general solution of $x^2y''-3xy'+3y = 2x^4e^x$.

First, we observe that the homogeneous equation is an Euler equation.

The solution has the form $y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$.

Substituting, we have $[r(r-1)-3r+3]x^r = 0$

I.e., $r^2-4r+3 = (r-1)(r-3) = 0$. So we have a solutions for r=1 and r=3, and

$$y_h = c_1x+c_2x^3.$$

Now, we can attempt to solve the non-homogeneous d.e. by variation of parameters.
I.e., we seek a solution in the form $y = xv_1+x^3v_2$

Then $y' = xv_1'+x^3v_2'+v_1+3x^2v_2$

We simplify by requiring that $\boxed{xv_1'+x^3v_2' = 0}$

Now $y'' = v_1' + 3x^2 v_2' + 6xv_2$

Substitution into the d.e. yields

$$x^2[v_1' + 3x^2 v_2' + 6xv_2] - 3x[v_1 + 3x^2 v_2] + 3[xv_1 + x^3 v_2] = 2x^4 e^x$$

I.e. $x^2[v_1' + 3x^2 v_2'] = 2x^4 e^x$

So we have two equations, after clearing common powers of x.

$$v_1' + 3x^2 v_2' = 2x^2 e^x$$
$$v_1' + x^2 v_2' = 0$$

Subtracting the second from the first,

$$2x^2 v_2' = 2x^2 e^x \text{ or } v_2' = e^x \text{ so } v_2 = e^x$$

Then from the second equation,

$$v_1' = -x^2 v_2' = -x^2 e^x$$

The integral is provided, so we are spared one integration by parts to have

$$v_1 = -(x^2 e^x - 2xe^x + 2e^x)$$

and finally, combine everything to have

$$\boxed{y = [c_1 - (x^2 e^x - 2xe^x + 2e^x)]x + [c_2 + e^x]x^3}.$$