

Ma 227 Homework 3 Solutions Fall 2009

Due 9/17/2009

9.1 - pg. 507 - 5,7

5. Express the given d.e. as a matrix system in normal form:

$$\begin{aligned}x' &= (\sin t)x + e^t y \\y' &= (\cos t)x + (a + bt^3)y\end{aligned}$$

We start by expressing right-hand sides of both equations as dot products:

$$(\sin t)x + e^t y = [\sin t, e^t] \cdot [x, y], \quad (\cos t)x + (a + bt^3)y = [\cos t] \cdot [a + bt^3]$$

Thus, by definition of the product of a matrix and column vector, the matrix form is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \sin t & e^t \\ \cos t & a + bt^3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

7. Express the given higher-order d.e. as a matrix system in normal form:

$$my'' + by' + ky = 0$$

Let $x_1 = y$

$$x_1' = y' = x_2$$

$$x_2' = y'' = \frac{-by'}{m} - \frac{ky}{m}$$

$$\Rightarrow x_2' = -\frac{bx_2}{m} - \frac{kx_1}{m}$$

so system is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

9.4 - pg. 530-31 - 1, 7, 11, 17, 21

1. Write the given system in the matrix form $x' = Ax + f$

$$\begin{aligned}x'(t) &= 3x(t) - y(t) + t^2 \\y'(t) &= -x(t) + 2y(t) + e^t\end{aligned}$$

Writing the matrix with the coefficients of x and y:

$$\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

Thus, this system becomes the equation in matrix form given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}.$$

7. Rewrite the given scalar equation as a first order system in normal form. Express the system in the in the matrix form $x' = Ax + f$.

$$\frac{d^4 w}{dt^4} + w = t^2.$$

Let $x_1(t) = w(t)$

$$x_1'(t) = \frac{dw}{dt} = x_2(t)$$

$$x_2'(t) = \frac{d^2 w}{dt^2} = x_3(t)$$

$$x_3'(t) = \frac{d^3 w}{dt^3} = x_4(t)$$

$$x_4'(t) = \frac{d^4 w}{dt^4} = t^2 - w = t^2 - x_1(t)$$

So system is:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

11. Write the given system as a set of scalar equations:

$$x' = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 5 \\ 0 & 5 & 1 \end{bmatrix} x + e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Multiplying the matrix values with the appropriate variable:

$$x_1' = 1(x_1)(t) + 0(x_2)(t) + 1(x_3)(t) + 1(e^t) + 0(t)$$

$$x_2' = -1(x_1)(t) + 2(x_2)(t) + 5(x_3)(t) + 0(e^t) + 1(t)$$

$$x_3' = 0(x_1)(t) + 5(x_2)(t) + 1(x_3)(t) + 0(e^t) + 0(t)$$

Simplifying:

$$x_1' = x_1(t) + x_3(t) + e^t$$

$$x_2' = -x_1(t) + 2x_2(t) + 5x_3(t) + t$$

$$x_3' = 5x_2(t) + x_3(t)$$

17. Determine whether the given vectors are linearly independent (LI) or linearly dependent (LD) on the interval $(-\infty, \infty)$

$$e^{2t} \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, e^{2t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, e^{3t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

By scalar multiplication:

$$\begin{bmatrix} e^{2t} \\ 0 \\ 5e^{2t} \end{bmatrix}, \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix}$$

To be linearly independent:

$$c_1 \begin{bmatrix} e^{2t} \\ 0 \\ 5e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} = 0$$

Writing as a system of equations:

$$c_1 + c_2 = 0$$

$$c_2 + c_3 = 0$$

$$5c_1 - c_2 = 0$$

$$\text{Solving} \Rightarrow c_1 = -c_2; c_2 = 0; c_3 = -c_2$$

$$\text{Therefore: } c_1 = c_2 = c_3 = 0$$

This tells us that the original set of vectors must be LI on the interval $(-\infty, \infty)$

21. The vectors $\vec{x}_1 = \begin{bmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ 2e^{3t} \end{bmatrix}$ are solutions to the

system

$$\vec{x}'(t) = A\vec{x}(t). \text{ Do they form a fundamental solution set?}$$

Take wronskian: $\begin{bmatrix} e^{-t} & e^t & e^{3t} \\ 2e^{-t} & 0 & -e^{3t} \\ e^{-t} & e^t & 2e^{3t} \end{bmatrix}$, determinant: $-2e^{3t} \neq 0$ then the system is

linearly independent thus,

$$\text{Fundamental Solution: } X(t) = \begin{bmatrix} e^{-t} & e^t & e^{3t} \\ 2e^{-t} & 0 & -e^{3t} \\ e^{-t} & e^t & 2e^{3t} \end{bmatrix}$$

General Solution: $X(t)c = c_1 \begin{bmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} + c_3 \begin{bmatrix} e^{3t} \\ -e^{3t} \\ 2e^{3t} \end{bmatrix}$

9.5 - pg. 541-52 - 11, 13, 19

11. Find a general solution to the system

$$A = \begin{bmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{bmatrix}$$

$$|A - rI| = \begin{bmatrix} -1-r & \frac{3}{4} \\ -5 & 3-r \end{bmatrix} = 0$$

$$r^2 - 2r + \frac{3}{4} = 0 \Rightarrow r = \frac{1}{2}, \frac{3}{2} \Rightarrow \text{eigenvalues}$$

$$(A - \frac{1}{2}r)u = \begin{bmatrix} -\frac{3}{2} & \frac{3}{4} \\ -5 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\frac{3}{2}u_1 + \frac{3}{4}u_2 = 0$$

$$\Rightarrow u_1 = \frac{1}{2}u_2 \Rightarrow u_2 = s; u_1 = \frac{1}{2}s$$

$$-5u_1 + \frac{5}{2}u_2 = 0$$

$$u = s \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow \text{eigenvector}$$

$$(A - \frac{3}{2}r)u = \begin{bmatrix} -\frac{5}{2} & \frac{3}{4} \\ -5 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\frac{5}{2}u_1 + \frac{3}{4}u_2 = 0$$

$$\Rightarrow u_1 = \frac{6}{20}u_2 \Rightarrow u_2 = s; u_1 = \frac{3}{10}s$$

$$-5u_1 + \frac{3}{2}u_2 = 0$$

$$u = s \begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix} \Rightarrow \text{eigenvector}$$

General Solution: $X(t)c = c_1 e^{\frac{1}{2}t} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{\frac{3}{2}t} \begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix}$

13. Find a general solution to the system

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} 1-r & 2 & 2 \\ 2 & -r & 3 \\ 2 & 3 & -r \end{vmatrix} = 0$$

$$\Rightarrow (1-r) \begin{vmatrix} -r & 3 \\ 3 & -r \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 2 & -r \end{vmatrix} + 2 \begin{vmatrix} 2 & -r \\ 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (1-r)(r^2 - 9) - 2(-2r - 6) + 2(6 + 2r) = (1-r)(r-2)(r+2) = 0$$

$$\Rightarrow (r+3)[(1-r)(r-3) + 8] = 0 \Rightarrow (r+3)(r-5)(r+1) = 0$$

The eigenvalues are:

$$r = -3, 5, -1$$

Eigenvector for:

$$r = -3$$

$$(A + 3I)u = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4u_1 + 2u_2 + 2u_3 = 0$$

$$2u_1 + 3u_2 + 3u_3 = 0$$

$$2u_1 + 3u_2 + 3u_3 = 0$$

Solving the system $\Rightarrow u_1 = 0; u_2 = -u_3;$

Setting $u_3 = s \Rightarrow u_2 = -s$

$$\Rightarrow u_1 = \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$r = 5$$

$$(A + 3I)u = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4u_1 + 2u_2 + 2u_3 = 0$$

$$2u_1 - 5u_2 + 3u_3 = 0$$

$$2u_1 + 3u_2 - 5u_3 = 0$$

$$\text{Solving the system} \Rightarrow u_1 = u_3, u_2 = u_3$$

$$\text{Letting } u_3 = s \Rightarrow u_1 = u_2 = u_3 = s$$

$$\Rightarrow u_2 = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$r = -1$$

$$(A + 3I)u = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2u_1 + 2u_2 + 2u_3 = 0$$

$$2u_1 + u_2 + 3u_3 = 0$$

$$2u_1 + 3u_2 + 1u_3 = 0$$

$$\text{Solving the system} \Rightarrow u_1 = -2u_2, u_3 = u_2$$

$$\text{Letting } u_2 = s \Rightarrow u_1 = -2s, u_3 = s$$

$$\Rightarrow u_3 = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the general solution for the system is:

$$x(t) = c_1 e^{-3t} u_1 + c_2 e^{-t} u_2 + c_3 e^{5t} u_3 = c_1 e^{-3t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

19. Find a fundamental matrix for the system $x'(t) = Ax(t)$ for the given matrix A

$$A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} -1-r & 1 \\ 8 & 1-r \end{vmatrix} = (-1-r)(1-r) - 8 = (r^2 - 9) = (r-3)(r+3)$$

The eigenvalues are : $r = 3, -3$

For $r = 3$

$$(A - rI)u = \begin{bmatrix} -4 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4u_1 + u_2 = 0$$

$$8u_1 - 2u_2 = 0$$

Solving the system $\Rightarrow u_2 = 4u_1$

Letting $u_1 = s \Rightarrow u_2 = 4s$

$$\Rightarrow u_1 = \begin{bmatrix} s \\ 4s \end{bmatrix} = s \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

For $r = -3$

$$(A - rI)u = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2u_1 + u_2 = 0$$

$$8u_1 + 4u_2 = 0$$

Solving the system $\Rightarrow u_2 = -2u_1$

Letting $u_1 = s \Rightarrow u_2 = -2s$

$$\Rightarrow u_2 = \begin{bmatrix} s \\ -2s \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, the general solution for the system is:

$$x(t) = c_1 e^{3t} u_1 + c_2 e^{-3t} u_2 = c_1 e^{3t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, the fundamental matrix for the system is:

$$\begin{bmatrix} e^{3t} & e^{-3t} \\ 4e^{3t} & -2e^{-3t} \end{bmatrix}$$