# Ma 227 Homework 10 Solutions Fall 2009
# Due 11/19/2009

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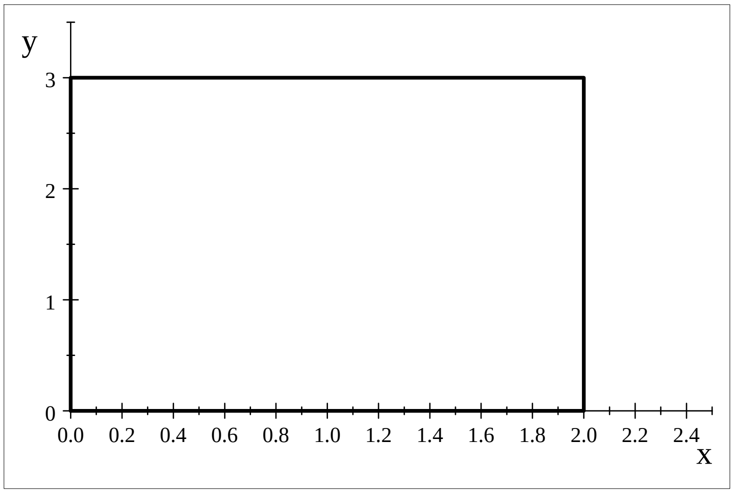
1) Evaluate the line integral bytwo methods: a)direclty, and b) using Green's Theorm.

$\oint_C xy^2dx + x^3dy$

$C$is the rectangle with vertices $(0,0)(2,0)(2,3)(0,3)$

$(0,0,2,0,2,3,0,3)$




a) directly: Let $C_1$ be the segment from $(0.0)$ to $(2,0)$, $C_2$ the segment from $(2,0)$ to $(2,3)$, $C_3$ the segment from $(2,3)$ to $(0,3)$, $C_4$ the segment from $(0,3)$ to $(0,0)$

$C_1 \to x = t, dx = dt; y = 0, dy = 0dt \quad 0 \le t \le 2$

$C_2 \to x = 2, dx = 0dt; \quad y = t, dy = dt \quad 0 \le t \le 3$

$C_3 \to x = 2 - t, dx = -dt; \quad y = 3, dy = 0 \quad 0 \le t \le 2$

$C_4 \to x = 0, dx = 0dt; \quad y = 3 - t, dy = -dt \quad 0 \le t \le 3$

Thus $\oint_C xydx + x^2y^3dy = \oint_{C_1+C_2+C_3+C_4} (xydx + x^2y^3dy)$

$= \int_0^2 0dt + \int_0^3 8dt + \int_0^2 -9(2-t)dt + \int_0^3 0dt = 0 + 24 - 18 + 0 = 6$

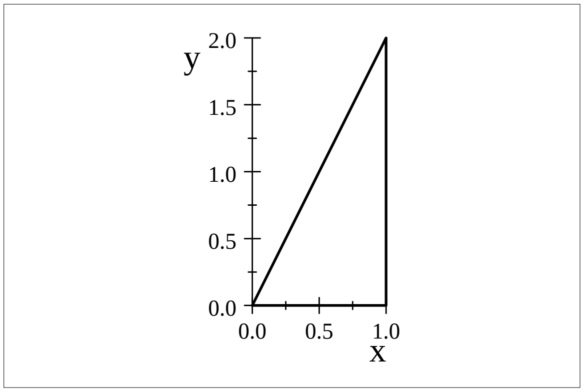
b) Using Greens Theorem:

$\oint_C xy^2dx + x^3dy = \int\int_D \left[ \frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(xy^2) \right] dA = \int_0^2 \int_0^3 (3x^2 + 2xy)dydx = \int_0^2 (9x^2 - 9x) = 24 -$

3) Evaluate the line integral bytwo methods: a)direclty, and b) using Green's Theorm.

$\oint_C xydx + x^2y^3dy$

$(0,0,1,0,1,2,0,0)$




a) directly: Let $C_1$ be the segment from $(0.0)$ to $(2,0)$, $C_2$ the segment from $(2,0)$ to $(2,3)$, $C_3$ the segment from $(2,3)$ to $(0,3), C_4$ the segment from $(0,3)$ to $(0,0)$

a) directly: Let $C_1$ be the segment from $(0.0)$ to $(1,0)$, $C_2$ the segment from $(1,0)$ to $(1,2)$ and $C_3$ the segment from $(1,2)$ to $(0,0)$.

$C_1 \to x = t, dx = dt; y = 0, dy = 0 \quad 0 \leq t \leq 1$

$C_2 \to x = 1, dx = 0dt; \quad y = t, dy = dt \quad 0 \leq t \leq 2$

$C_3 \to$The equation of this line is $y = 2x$. Thus $x = t, dx = dt; y = 2t, dy = 2dt \quad t : 1 \to 0$

$$\oint_C xydx + x^2y^3dy = \int_{C1} xydx + x^2y^3dy + \int_{C2} xydx + x^2y^3dy + \int_{C3} xydx + x^2y^3dy$$

$$= \int_0^1 0dt + \int_0^2 t^3dt + \int_1^0 t(2t)dt + \int_1^0 t^2(8t^3)(2dt) = \int_0^2 t^3dt + \int_1^0 (2t^2)dt + 2\int_1^0 (8t^5)dt = \frac{2}{3}$$

b) Using Greens Theorem:

$$\oint_C xydx + x^2y^3dy = \int\int_D [\partial/\partial x\ (x^2y^3) - \partial/\partial y\ (xy)]dA$$
$$= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx$$
$$= 2/3$$

5)

$P(x,y) = x^4y^5; Q(x,y) = -x^7y^6$

$C \to x^2 + y^2 = 1$

Since $C$ is a circle of radius 1, we parametrize $C$ as

$x = \cos\theta, dx = -\sin\theta d\theta \qquad 0 \leq \theta \leq 2\pi$

$y = \sin\theta, dy = \cos\theta d\theta$

$\oint_C x^4y^5dx - x^7y^6dy =$

$\int_0^{2\pi} \cos^4\theta \sin^5\theta(-\sin\theta)d\theta - \int_0^{2\pi} \cos^7\theta \sin^6\theta(\cos\theta)d\theta = -\int_0^{2\pi} (\cos^4\theta \sin^6\theta + \cos^8\theta \sin^6\theta)d\theta = -\frac{29}{1024}\pi$

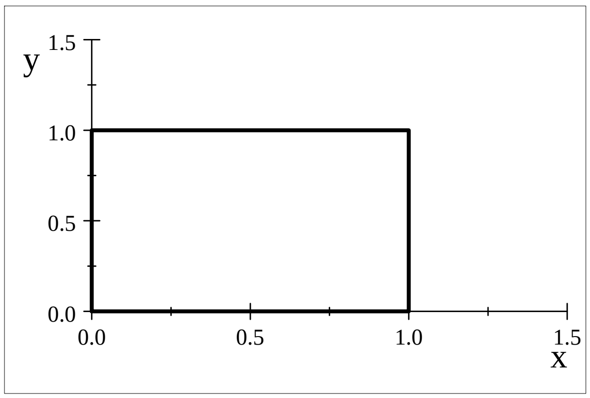
Note that it was necessary to simply the integral in order to have SNB evaluate it.

Double Integral:

$x^2 + y^2 = 1$
$x$ goes from $-1$ to $1$
$y$ goes from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$$\int\int_D [\partial/\partial x\ (Q) - \partial/\partial y\ (P)]dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(-7\ x^6y^6 - 5x^4y^4\right)dydx = -\tfrac{29}{1024}\pi$$

7) Use Green's Theorm to evaluate the line integral along the given positivley oriented curve.
$\int_C e^y dx + 2xe^y dy$, $C$ is the square with sides $x = 0, x = 1, y = 0$ and $y = 1$
$(0,0,0,1,1,1,1,0)$




The region $D$ enclosed by $C$ is $[0,1] \times [0,1]$, *so*
$$\int_C e^y dx + 2xe^y dy = \int\int_D \left[\tfrac{\partial}{\partial x}(2xe^y) - \tfrac{\partial}{\partial y}(e^y)\right]dA = \int_0^1\int_0^1 (2e^y - e^y)dydx$$
$$= \int_0^1 dx \int_0^1 (e^y)dy = (1)(e^1 - e^0) = e - 1$$

13) $F(x,y) = \left\langle \sqrt{x} + y^3, x^2 + \sqrt{y} \right\rangle$,
$C$ consists of the arc of the curve $y = \sin x$ from $(0,0)$ to $(\pi,0)$ and the line segment from $(\pi,0)$ to $(0,0)$
$C$ is traversed clockwise, so $-C$ gives the positive orientation
$$\int_C F \cdot dr = -\int_{-C}(\sqrt{x} + y^3)dx + \left(x^2 + \sqrt{y}\right)dy = -\int\int_D \left[\tfrac{\partial}{\partial x}\left(x^2 + \sqrt{y}\right) - \tfrac{\partial}{\partial y}(\sqrt{x} + y^3)\right]dA$$
$$= -\int_0^\pi\int_0^{\sin x}(2x - 3y^2)dydx = -\int_0^\pi [2xy - y^3]_{y=0}^{y=\sin x}dx$$
$$= -\int_0^\pi (2x\sin x - \sin^3 x)dx = -\int_0^\pi (2x\sin x - (1-\cos^2 x)\sin x)dx \qquad [\text{Use integration by parts}]$$
$$= -\left[2\sin x - 2x\cos x + \cos x - \tfrac{1}{3}\cos^3 x\right]_0^\pi = -\left(2\pi - 2 + \tfrac{2}{3}\right) = \tfrac{4}{3} - 2\pi$$

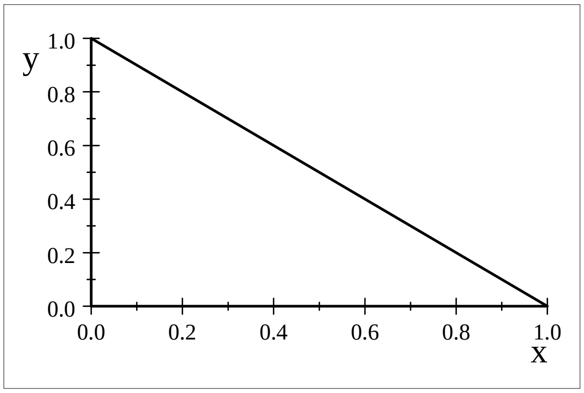
15) $F(x,y) = \langle e^x + x^2y, e^y - xy^2\rangle$,
$C$ is the circle $x^2 + y^2 = 25$ orientated clockwise.
$C$ is traversed clockwise, so $-C$ gives the positive orientation
$$\int_C F \cdot dr = -\int_{-C}(e^x + x^2y)dx + (e^y - xy^2)dy = -\int\int_D \left[\tfrac{\partial}{\partial x}(e^y - xy^2) - \tfrac{\partial}{\partial y}(e^x + x^2y)\right]dA$$
$$= -\int\int_D (-y^2 - x^2)dA = \int\int_D (x^2 + y^2)dA = \int_0^{2\pi}\int_0^5 (r^2)rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 (r^3)dr = 2\pi[\tfrac{1}{4}r^4]_0^5 = \tfrac{625}{2}\pi$$

17) Use Green's Theorm to find the work done by the force $F(x,y) = x(x+y)\vec{i} + xy^2\vec{j}$ in moving a particle from the origin along the x-axis to $(1,0)$ then along the line segment to $(0,1)$, and the back to the origin along the y-axis.

The path is shown below.

$(0,0,1,0,0,1,0,0)$




The line joining $(1,0)$ to $(0,1)$ has equation $y = 1-x$. Thus

By Greens Theorem,

Work= $\int_C F.dr = \oint_C x(x+y)dx + xy^2dy = \int\int_D (y^2 - x)dydx$ where $C$ is the path described in the question and D is the triangle bounded by C.

Work $= \int_0^1 \int_0^{1-x} (y^2 - x)dydx = -\tfrac{1}{12}$

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5) Evaluate $\int\int_s yzdS \qquad S: x = uv, \qquad y = u+v \qquad z = u-v$

$$u^2 + v^2 \leq 1$$

$\int\int_s yzdS = \int\int_D f(r(u,v))|r_u \times r_v|dA$

define: $\vec{r}(u,v) = (uv)\,\vec{i} + (u+v)\,\vec{j} + (u-v)\,\vec{k}$

$\vec{r}_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) = (v,1,1)$

$\vec{r}_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) = (u,1,-1)$

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} i & j & k \\ v & 1 & 1 \\ u & 1 & -1 \end{bmatrix} = -2\,\vec{i} + (u+v)\,\vec{j} + (v-u)\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{2^2 + (u+v)^2 + (u-v)^2} = \sqrt{4 + 2u^2 + 2v^2}$$

$\int\int_s yzdS = \int\int_{u^2+v^2\le 1}\Big((u+v)(u-v)\sqrt{4+2u^2+2v^2}\Big)dudv$
$\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{-\sqrt{1-u^2}}(u^2-v^2)\sqrt{4+2u^2+2v^2})dudv$
or with polar coordinates $u = r\cos\theta, v = r\sin\theta$:
$\int_0^{2\pi}\int_0^1\Big(\cos 2\theta\sqrt{4+2r^2}\Big)rdrd\theta = 0$

7) $\int\int_s x^2yzdS$ $S$ is the part of the plane $z = 1+2x+3y$ that lies above $[0,3]\times[0,2]$
Then: $0 \le x \le 3$ $0 \le y \le 2$
$\frac{\partial z}{\partial x} = 2$ $\frac{\partial z}{\partial y} = 3$
$\int\int_s x^2yzdS = \int\int_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\ dA$
$\int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{4+9+1}\,dydx$
$\int_0^3\int_0^2\sqrt{14}\,(x^2y+2x^3y+3x^2y^2)dydx = 171\sqrt{14}$

9) $\int\int_s yzdS$ $S$ is the part of the plane $x+y+z = 1$ that lies in the first octant
$z = 1-(x+y)$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -1$
$\int\int_D y(1-x-y)\sqrt{(-1)^2+(-1)^2+1^2}\,dA$
$\int_0^1\int_0^{1-x}(\sqrt{3}\,y(1-x-y))dydx = \frac{1}{24}\sqrt{3}$

15) $\int\int_s(x^2z+y^2z)dS$
$S$ is the hemisphere $x^2+y^2+z^2 = 4,\ z \ge 0$

Using sphyrical coordinates coordinates:
$\vec{r}(\phi,\theta) = 2\sin\phi\cos\theta\vec{i}+2\sin\phi\sin\theta\vec{j}+2\cos\phi\vec{k}$
$|\vec{r}_\theta\times\vec{r}_z| = 4\sin\phi$
$\int\int_s(x^2z+y^2z)dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)d\phi d\theta = [16\pi\sin^4\phi]_0^{\frac{\pi}{2}} = 16\pi$