

# Ma 227 Homework Assignment 11

## Fall 2009 Due 12/3/09

Section 13.6

958 # 19, 21, 23, 27

Evaluate  $\iint_S \vec{F} \cdot d\vec{S}$  for the given vector field  $\vec{F}$  and the oriented surface  $S$ .

19.  $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

$S$  is the part of the paraboloid  $z = 4 - x^2 - y^2$  that lies above

$0 \leq x \leq 1, 0 \leq y \leq 1$  and has upward direction

$S : \vec{r}(u, v) = \vec{r}(x, y) = (x, y, 4 - x^2 - y^2)$

$\vec{r}_x = (1, 0, -2x)$

$\vec{r}_y = (0, 1, -2y)$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x, 2y, 1)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) dA$$

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = (xy, yz, zx) \cdot (2x, 2y, 1) = xz + 2x^2y + 2y^2z$$

$$\int_0^1 \int_0^1 (xz + 2x^2y + 2y^2z) dx dy$$

$$\int_0^1 \int_0^1 (x(4 - x^2 - y^2) + 2x^2y + 2y^2(4 - x^2 - y^2)) dx dy = \frac{713}{180}$$

21.  $\vec{F}(x, y, z) = xze^y \vec{i} - xze^y \vec{j} + z \vec{k}$

$S$  is the part of the plane  $x + y + z = 1$  in the first octant in the downward direction.

$$\iint_S \vec{F} \cdot d\vec{S} \quad \vec{r} = (x, y, 1 - x - y)$$

$\vec{r}_x = (1, 0, -1)$

$\vec{r}_y = (0, 1, -1)$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$$

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = (xze^y, -xze^y, z) \cdot (1, 1, 1) = z$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D z dA = -\int_0^1 \int_0^{1-x} (1 - x - y) dy dx = -\frac{1}{6}$$

(negative sign to indicate downward direction)

23.  $\vec{F}(x, y, z) = x\vec{i} - z\vec{j} + y\vec{k}$

$S$  is the part of the sphere  $x^2 + y^2 + z^2 = 4$  in the first octant, with orientation toward the origin.

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= -\iint_D \left[ x\left(\frac{1}{2}\right)(4-x^2-y^2)^{-\frac{1}{2}}(-2x) - (-z)\frac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}}(-2y) + y \right] dA \\
&= -\iint_D \left( \frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2} \cdot \frac{y}{\sqrt{4-x^2-y^2}} + y \right) dA = -\iint_D x^2(4-(x^2+y^2))^{-\frac{1}{2}} dA \\
&= -\int_0^{\frac{\pi}{2}} \int_0^2 (r \cos \theta)^2 (4-r^2)^{-\frac{1}{2}} r dr d\theta = -\int_0^{\frac{\pi}{2}} r \cos \theta d\theta \int_0^2 r^3 (4-r^2)^{-\frac{1}{2}} dr \quad [\text{Let} \\
&u = 4-r^2 \Rightarrow r^2 = 4-u \& -\frac{1}{2} du = r dr] \\
&= -\int_0^{\frac{\pi}{2}} \left( \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta \int_4^0 -\frac{1}{2} (4-u)(u)^{-\frac{1}{2}} du = -\left[ \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{2}} \left( -\frac{1}{2} \right) \left[ 8\sqrt{u} - \frac{2}{3} u^{\frac{3}{2}} \right]_4^0 =
\end{aligned}$$

$$27. \vec{F}(x, y, z) = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$S$  is the boundary of the solid half cylinder  $0 \leq z \leq \sqrt{1-y^2}$ ,  $0 \leq x \leq 2$

Here  $S$  consists of four surfaces:  $S_1$ , the top surface (a portion of the circular cylinder ( $y^2 + z^2 = 1$ )),  $S_2$ , the bottom surface (a portion of the  $xy$ -plane);  $S_3$ , the front half disk in the plane  $x = 2$ , and  $S_4$ , the back half disk in the plane  $x = 0$

On  $S_1$  the surface is  $z = \sqrt{1-y^2}$  for  $0 \leq x \leq 2, -1 \leq y \leq 1$  with upward direction:

$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot d\vec{S} &= \int_0^2 \int_{-1}^1 \left[ -x^2(0) - y^2 \left( -\frac{y}{\sqrt{1-y^2}} \right) + z^2 \right] dy dx = \int_0^2 \int_{-1}^1 \left[ \left( -\frac{y^3}{\sqrt{1-y^2}} + 1 - y^2 \right) \right] dy dx \\
&= \int_0^2 \left[ \sqrt{1-y^2} + \frac{1}{3}(1-y^2)^{\frac{3}{2}} + y - \frac{1}{3}y^3 \right]_{y=-1}^{y=1} dx = \int_0^2 \frac{4}{3} dx = \frac{8}{3}
\end{aligned}$$

On  $S_2$  the surface is  $z = 0$  with downward orientation:

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \int_0^2 \int_{-1}^1 -z^2 dy dx = \int_0^2 \int_{-1}^1 (0) dy dx = 0$$

On  $S_3$  the surface is  $x = 2$  for  $-1 \leq y \leq 1, 0 \leq z \leq \sqrt{1-y^2}$ , oriented in the positive  $x$ -direction. Regarding  $y$  and  $z$  as parameters, we have:

$$r_y \times r_z = \vec{i} \text{ and}$$

$$\iint_{S_3} \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2 dz dy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 4 dz dy = 4A(S_3) = 2\pi$$

On  $S_4$  the surface is  $x = 0$  for  $-1 \leq y \leq 1, 0 \leq z \leq \sqrt{1-y^2}$ , oriented in the negative  $x$ -direction. Regarding  $y$  and  $z$  as parameters, we have:

$$-(r_y \times r_z) = \vec{i} \text{ and}$$

$$\iint_{S_4} \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2 dz dy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 0 dz dy = 0$$

Thus:

$$\iint_s \vec{F} \cdot d\vec{S} = \frac{8}{3} + 0 + 2\pi + 0 = 2\pi + \frac{8}{3}$$