Ma 227 Homework 12 Solutions Fall 2009 Due 12/10/2009 (Not to be turned in)

Page 964 3, 5, 9, Stokes' Theorem
Section 13.7
3. $\vec{F}(x,y,z) = x^2e^{yz}\vec{i} + y^2e^{xz}\vec{j} + z^2e^{xy}\vec{k}$, S is the hemisphere $x^2+y^2+z^2 = 0, z \geq 0$ oriented upwards.

The boundary curve C is the circle $x^2+y^2 = 4, z = 0$ oriented in the counterclockwise direction. Thus using polar coordinates we have

$$r(t) = 2\cos t\vec{i} + 2\sin t\vec{j}$$
$$r'(t) = -2\sin t\vec{i} + 2\cos t\vec{j}$$

Also

$$\vec{F}(x,y,z) = x^2e^{yz}\vec{i} + y^2e^{xz}\vec{j} + z^2e^{xy}\vec{k}$$

so

$$\vec{F}(\vec{r}(t)) = (2\cos t)^2e^{(2\sin t)(0)}\vec{i} + (2\sin t)^2e^{(2\cos t)(0)}\vec{j} + 0^2e^{(2\cos t)(2\sin t)}\vec{k}$$

now by Stoke's theorem:

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_c \vec{F}\cdot d\vec{r} = \int_0^{2\pi} F(r(t)) \cdot r'(t)dt$$
$$= \int_0^{2\pi}(-8\cos^2 t\sin t + 8\sin^2 t\cos t)dt = 0$$

5. C is the square in the plane $z = -1$. By (3)

$$\iint_{S_1}\operatorname{curl}\vec{F}d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = \iint_{S_2}\operatorname{curl}\vec{F}d\vec{S}$$

where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube.

$$\operatorname{curl}\vec{F} = (x^2z)\vec{i} + (xy-2xyz)\vec{j} + (y-xz)\vec{k}$$

For S_2 we choose $\vec{n} = \vec{k}$ so that C has the same orientation for both surfaces. Then

$$\operatorname{curl}\vec{F}\cdot\vec{n} = y - xz = x + y$$

since $z = -1$. Thus

$$\iint_{S_2}\operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{-1}^{1}\int_{-1}^{1}(x+y)dxdy = 0$$

so

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = 0$$

9.

$$\text{curl}\vec{F} = (xe^{xy} - y)\vec{i} - (ye^{xy} - y)\vec{j} - (2z - z)\vec{k}$$

Take the surface S to be the disk $x^2 + y^2 \leq 16, z = 5$. Since C is oriented clockwise (from above), we orient S upward. Then $n = k$ and $curlF \cdot n = 2z - z$ on S were $z = 5.$ Thus

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_s \text{curl}\vec{F} \cdot nd\vec{S} = \iint_D [2z - z]dS$$

$$\iint_D (10 - 5)dS$$

$$= 5(Area)_S = 5(\pi \cdot 4^2) = 80\pi$$

Page 971 3, 5, 7, 9, 13

Section 13.8

3.Verify the divergence theorm is true for the vector field F on the region E.

$\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}. E$ is the solid cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1.$

$div\vec{F} = x + y + z$

so using cylindrical coordinates,

$$\int\int\int_E \text{div} FdV = \int_0^{2\pi}\int_0^1\int_0^1 (r\cos\theta + r\sin\theta + z)rdzdrd\theta = \frac{1}{2}\pi(\text{using SNB})$$

Let S_1 be the top of the cylinder, S_2 be the bottom, and S_3 be the vertical edge

$S_1 : \vec{n} = \vec{k}; z = 1; \vec{F} = xy\vec{i} + y\vec{j} + x\vec{k}$ Then

$\int\int_{S_1} \vec{F} \cdot d\vec{S} = \int\int_{S_1} xdA = \int_0^{2\pi}\int_0^1 (r\cos\theta)rdrd\theta = 0$

$S_2 : \vec{n} = -\vec{k}; z = 0; \vec{F} = xy\vec{i}$ Then $\int\int_{S_2} \vec{F} \cdot d\vec{S} = \int\int_{S_2} 0dS = 0$

S_3 : We use cylindrical coordinates to parametrize this surface so that

$\vec{r}(\theta, z) = \cos\theta\vec{i} + \sin\theta\vec{j} + z\vec{k}$

and $0 \leq \theta \leq 2\pi, 0 \leq z \leq 1.$ Also $\vec{F}(\theta, z) = \cos\theta\sin\theta\vec{i} + z\sin\theta\vec{j} + z\vec{k}$

$\vec{r}_\theta \times \vec{r}_z = \cos\theta\vec{i} + \sin\theta\vec{j}$

Then

$\int\int_{S_3} \vec{F} \cdot d\vec{S} = \int\int_D F \cdot (\vec{r}_\theta \times \vec{r}_z)dA = \int_0^{2\pi}\int_0^1 (\cos^2\theta\sin\theta + z\sin^2\theta)dzd\theta = \frac{1}{2}\pi$

Therefore

$\int\int_S F.dS = 0 + 0 + \frac{1}{2}\pi = \frac{1}{2}\pi$

5. $\operatorname{div}\vec{F} = \partial/\partial x(e^x \sin y) + \partial/\partial y(e^x \cos y) + \partial/\partial z(yz^2) = e^x \sin y - e^x \sin y + 2yz = 2yz$

by the Divergence theorem,

$$\int\int_S \vec{F} \bullet d\vec{S} = \int\int\int_E \operatorname{div} F dV = \int_0^1 \int_0^1 \int_0^2 2yzdzdydx = 2$$

7. $\vec{F}(x,y,z) = 3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k}$. S is the surface of the soild bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

$\operatorname{div} F = 3y^2 + 0 + 3z^2 = 3y^2 + 3z^2$

so using cylindrical coordinates where $y = r\cos\theta, z = r\sin\theta$ and $x = x$

$\int\int_S \vec{F} \bullet d\vec{S} = \int\int\int_E (3y^2 + 3z^2)dV = \int_0^{2\pi} \int_0^1 \int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)rdxdrd\theta = \frac{9}{2}\pi$

9. $\vec{F}(x,y,z) = xy\sin z\vec{i} + \cos(xz)\vec{j} + y\cos z\vec{k}$. S is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$\operatorname{div} F = y\sin z + 0 - y\sin z$ so by Divergence Therom

$\int\int_S \vec{F} \bullet d\vec{S} = \int\int\int_E 0dV = 0$

13. $\vec{F}(x,y,z) = 4x^3z\vec{i} + 4y^3ze^z\vec{j} + 3z^4\vec{k}$. S is the spherewith radius R and center the origin.

$\operatorname{div} F = 12x^2z + 12y^2z + 12z^3$ so using spherical coordinates

$\int\int_S \vec{F} \bullet d\vec{S} = \int\int\int_E 12z(x^2 + y^2 + z^2)dV = \int_0^{2\pi} \int_0^{\pi} \int_{-1}^{R} 12(\rho\cos\phi)(\rho^2)\rho^2 \sin\phi d\rho d\phi d\theta =$

$12\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\cos\phi d\phi \int_0^R \rho^5 d\rho = 12(2\pi)[\frac{1}{2}\sin^2\phi]_0^{\pi}\left[\frac{1}{6}\rho^6\right]_0^R = 0$