

Ma 227 Homework 6 Solutions Fall 2009

Due 10/22/2009

12.4-pg. 856 #3, 5, 7, 9, 11, 15, 19, 21, 23, 27

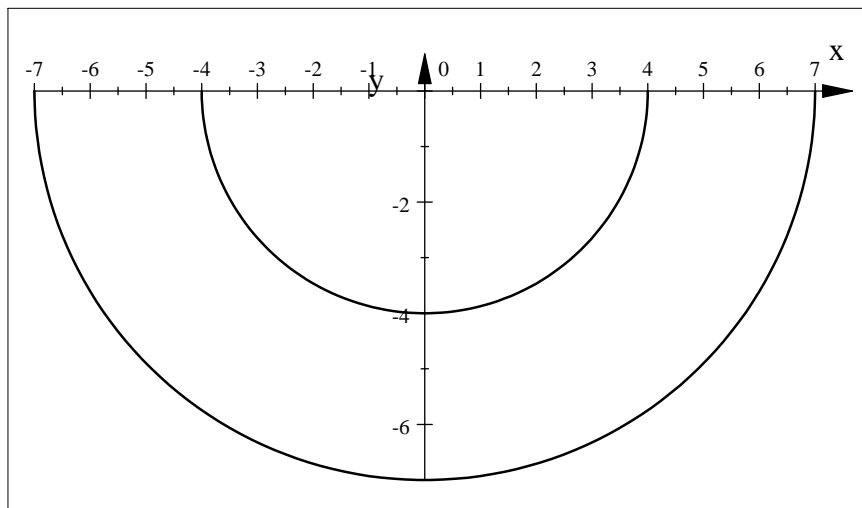
3. The region R is more easily described by the rectangular coordinates:
 $R = \{(x,y) \mid -2 \leq x \leq 2, 2 \leq y \leq x\}$

$$\text{Thus } \iint_R f(x,y) dA = \int_{-2}^2 \int_x^2 f(x,y) dy dx$$

5. The region R is more easily described by polar coordinates:
 $R = \{(r,\theta) \mid 2 \leq r \leq 5, 0 \leq \theta \leq 2\pi\}$

$$\text{Thus } \iint_R f(x,y) dA = \int_0^{2\pi} \int_2^5 f(r \cos \theta, r \sin \theta) r dr d\theta$$

7. $\int_{\pi}^{2\pi} \int_4^7 r dr d\theta$ represents the area of the region $R = \{(r,\theta) \mid 4 \leq r \leq 7, \pi \leq \theta \leq 2\pi\}$,
 the lower half of a ring as shown below.
 $r = 4$



$$\int_{\pi}^{2\pi} \int_4^7 r dr d\theta = \int_{\pi}^{2\pi} \left[\frac{r^2}{2} \right]_4^7 d\theta = \pi \left(\frac{1}{2} \right) (49 - 16) = \frac{33\pi}{2}$$

9. $\iint_D xy dA$, where D is the disk with center at the origin and radius 3.
 D can be described as $D = \{(r,\theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$

\Rightarrow

$$\begin{aligned} \iint_D xy dA &= \int_0^{2\pi} \int_0^3 (r \cos \theta)(r \sin \theta) r dr d\theta = \int_0^{2\pi} \int_0^3 r^3 \cos \theta \sin \theta dr d\theta \\ &= \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^3 \cos \theta \sin \theta dr d\theta = \frac{81}{4} \left(\frac{1}{2} \sin^2 \theta \right)_0^{2\pi} = 0 \end{aligned}$$

11. $\iint_R \cos(x^2 + y^2) dA$ where R is the region that lies above the x -axis within the circle $x^2 + y^2 = 9$

D can be described as $D = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq \pi\}$

15. Find the volume under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$.

$$V = \iint_{x^2+y^2 \leq 4} \sqrt{x^2 + y^2} dA = \int_0^{2\pi} \int_0^2 \sqrt{r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_0^2 r^2 dr = [\theta]_0^{2\pi} \left[\frac{1}{3} r^3 \right]_0^2 = 2\pi \left(\frac{8}{3} \right) = \frac{16\pi}{3}$$

19. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So:

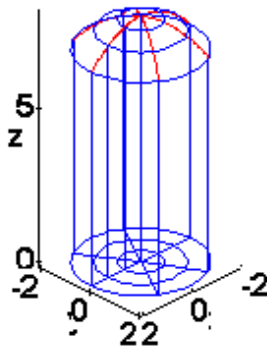
$$V = \iint_{x^2+y^2 \leq \frac{1}{2}} (\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}) dA = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} (\sqrt{1 - r^2} - r) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{1}{\sqrt{2}}} (r\sqrt{1 - r^2} - r^2) dr = [\theta]_0^{2\pi} \left[-\frac{1}{3} (1 - r^2)^{\frac{3}{2}} - \frac{1}{3} r^3 \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \pi \left(-\frac{1}{3} \right) \left(\frac{1}{\sqrt{2}} - 1 \right) = \frac{\pi}{3} (2 - \sqrt{2})$$

21. Use polar coordinates to find the volume inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$.

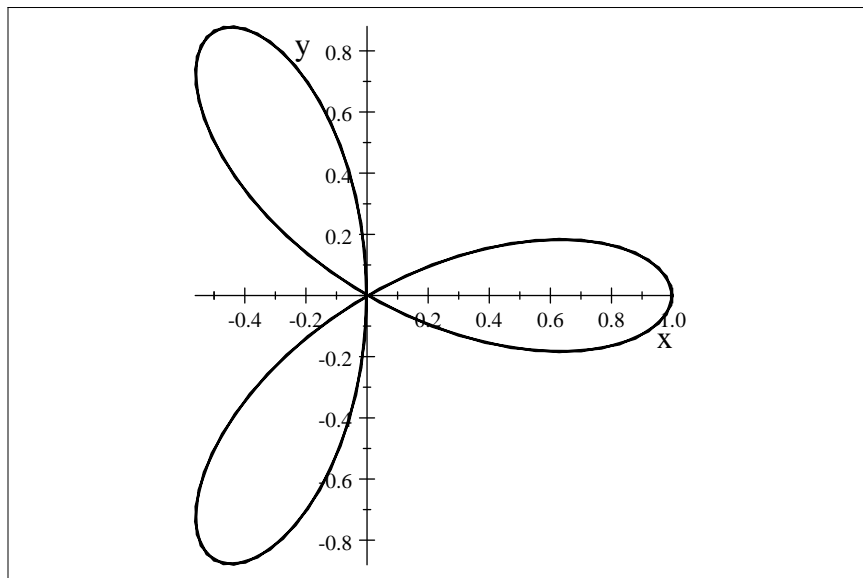
SOLUTION



The ellipsoid intersects the x, y -plane in the circle $x^2 + y^2 = 16$. Thus, our region is bounded by the circle $x^2 + y^2 = 4$. So, in polar coordinates we have the equation $r = 2$. Next, we can solve the equation of the ellipsoid $4x^2 + 4y^2 + z^2 = 64$ for z , i.e., $z = \pm 2\sqrt{-x^2 - y^2 + 16}$ which can be rewritten in polar coordinates as $z = \pm 2\sqrt{16 - r^2}$. The volume of the solid can now be written as:

$$2 \int_0^{2\pi} \int_0^2 (2\sqrt{16 - r^2}) r dr d\theta = -64\sqrt{3}\pi + \frac{512}{3}\pi$$

23. Use a double integral to find the area of one loop of the rose $r = \cos 3\theta$.
 $r = \cos 3\theta$



$r = 0 \Rightarrow \cos 3\theta = 0$ or $3\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{6}$. Thus for the loop in the first and fourth quadrants we have $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. Using symmetry we get

$$\text{Area} = 2 \int_0^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r dr d\theta = \frac{\pi}{12}$$

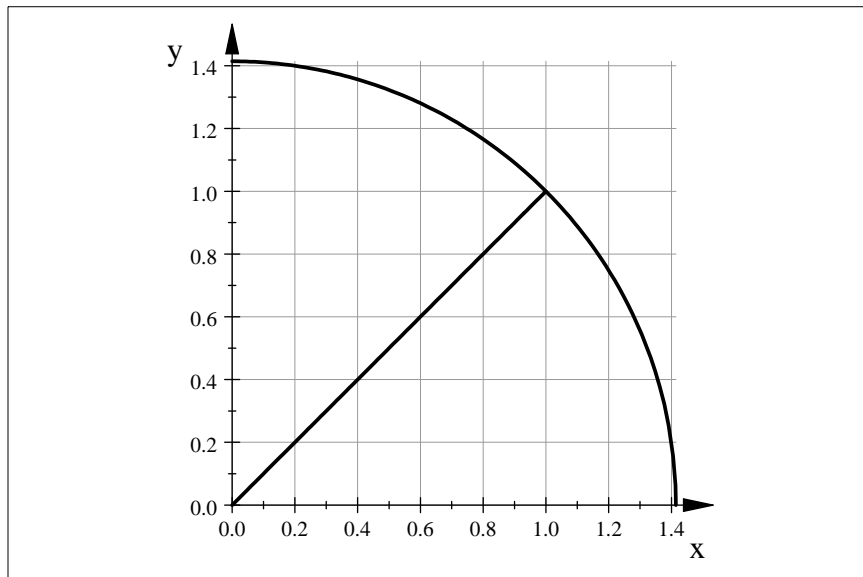
27. Evaluate

$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x + y) dx dy$$

by converting to polar coordinates.

The region of integration is the part of the circle of radius $\sqrt{2}$ centered at the origin in the first quadrant below the line $y = x$.

$$\sqrt{2 - y^2}$$



Thus

$$\begin{aligned} \int_0^1 \int_y^{\sqrt{2-x^2}} (x+y) dy dx &= \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} (r \cos \theta + r \sin \theta) r dr d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3} \right]_0^{\sqrt{2}} (\cos \theta + \sin \theta) d\theta \\ &= \frac{2\sqrt{2}}{3} [\sin \theta - \cos \theta]_0^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{3} \left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 0 + 1 \right] = \frac{2\sqrt{2}}{3} \end{aligned}$$

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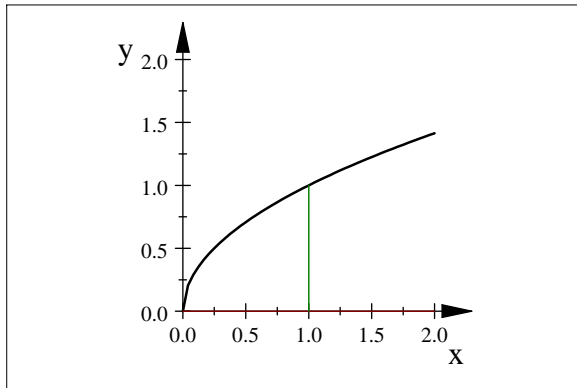
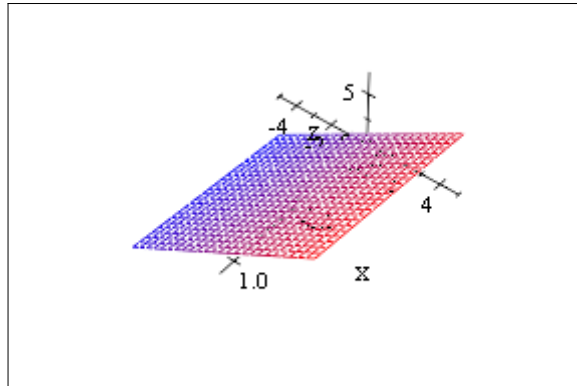
5.

$$\begin{aligned} \int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} z e^y dx dz dy &= \int_0^3 \int_0^1 (\sqrt{1-z^2}) z e^y dz dy \\ &= \int_0^3 \left[-\frac{(1-z^2)^{\frac{3}{2}}}{3} \right]_0^1 e^y dy \\ &= \frac{1}{3} \int_0^3 e^y dy = \frac{1}{3} (e^3 - 1) \end{aligned}$$

7. Evaluate $\iiint_E 2x dV$ where $E = \{(x, y, z) \mid 0 \leq z \leq y, 0 \leq y \leq 2, 0 \leq x \leq \sqrt{4-y^2}\}$.

$$\begin{aligned} &\iiint_E 2x dV \\ &= \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^y 2x dz dx dy = \int_0^2 \int_0^{\sqrt{4-y^2}} 2xy dx dy = \int_0^2 (4-y^2)y dy = [2y^2 - \frac{1}{4}y^4]_0^2 = 4 \end{aligned}$$

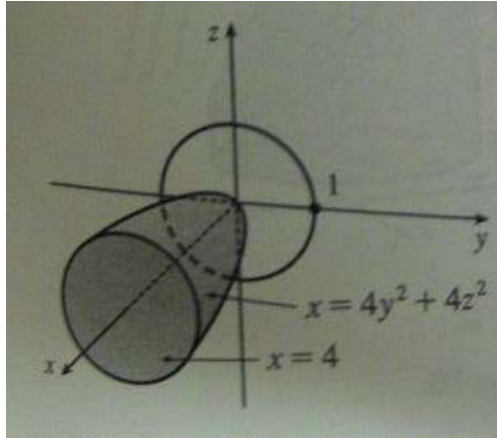
9. Evaluate $\iiint_E 6xy dV$ where E lies under the plane $z = 1 + x + y$ and above the region in the xy -plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$



Here $E = \{(x, y, z) | 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, 0 \leq z \leq 1 + x + y\}$, so

$$\begin{aligned} \iiint_E 6xy dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xyz dz dy dx = \int_0^1 \int_0^{\sqrt{x}} [6xyz]_{z=0}^{z=1+x+y} dy dx = \int_0^1 \int_0^{\sqrt{x}} 6xy(x+y+1) dy dx \\ &= \int_0^1 [3xy^2 + 3x^2y^2 + 2xy^3]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 (3x^2 + 3x^3 + 2x^{\frac{5}{2}}) dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{\frac{7}{2}} \right]_0^1 = \frac{65}{28} \end{aligned}$$

15. $\iiint_E x dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$

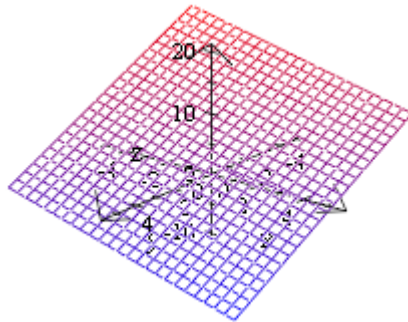


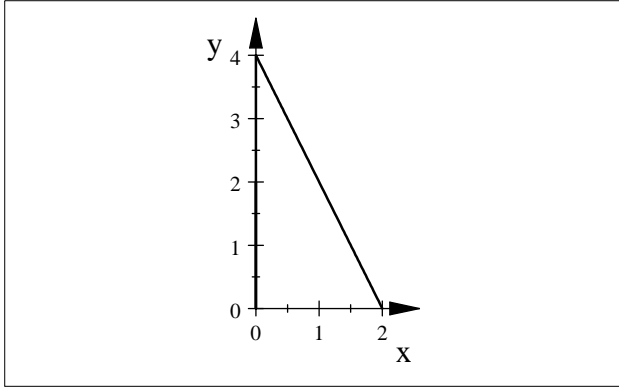
The projection on the yz -plane is the disk $y^2 + z^2 \leq 1$. Using polar coordinates $y = r \cos \theta$ and $z = r \sin \theta$ we get:

$$\begin{aligned} \iiint_E x dV &= \iint_D \left[\int_{4y^2+4z^2}^4 x dx \right] dA = \frac{1}{2} \iint_D [4^2 - (4y^2 + 4z^2)^2] dA = 8 \int_0^{2\pi} \int_0^1 [1 - r^4] r dr d\theta \\ &= 8 \int_0^{2\pi} d\theta \int_0^1 [r - r^5] dr = 8(2\pi) \left[\frac{1}{2} r^2 - \frac{1}{6} r^6 \right]_0^1 = \frac{16}{3} \pi \end{aligned}$$

17. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x + y + z = 4$.

The plane $2x + y + z = 4$ intersects the xy -plane in the line $y = 4 - 2x$,





$$\text{So } E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 4 - x, 0 \leq z \leq 4 - 2x - y\}$$
$$\Rightarrow V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz dy dx = \frac{16}{3}$$