Ma 227 Review for Systems of DEs

Matrices

Basic Properties

Addition and subtraction:

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then

$$A \pm B = [a_{ij} \pm b_{ij}]_{m\times n}$$

Example:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 4 & 7 \\ -1 & -2 & -6 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+6 & -2+4 & 3+7 \\ 0-1 & -1-2 & 6-6 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 10 \\ -1 & -3 & 0 \end{bmatrix}$$

Scaler Multiplication:

Let k be a scalar and A a matrix of real numbers of order $m \times n$. Then

$$kA = [k \cdot a_{ij}]_{m\times n}$$

Example:

$$5\begin{bmatrix} -1 & 0 & 5 & 7 \\ 2 & -8 & 4 & 22 \\ -7 & 1 & 0 & 6 \\ 8 & 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 25 & 35 \\ 10 & -40 & 20 & 110 \\ -35 & 5 & 0 & 30 \\ 40 & 15 & -15 & 20 \end{bmatrix}$$

Some Properties of Addition and Scalar Multiplication

Theorem

Let A, B and C be conformable $m \times n$ matrices whose entries are real numbers, and k and p arbitrary scalars. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$
3. There is an $m \times n$ matrix 0 such that $0 + A = A$ for each A.
4. For each A there is an $m \times n$ matrix $-A$ such that $A + (-A) = 0$.

5. $k(A+B) = kA + kB$
6. $(k+p)A = kA + pA$
7. $(kp)A = k(pA)$.

(4) Note that $(-1)A = [-a_{ij}]_{m\times n} \quad \Rightarrow A + (-1)A = 0_{m\times n}$

Remark: We denote $(-1)A$ by $-A$.

The Transpose of a Matrix

If A is an $m \times n$ matrix, the transpose of A, denoted A^T, is the $n \times m$ matrix whose entry a_{st} is the same as the entry a_{ts} in the matrix A. Thus one gets the transpose of A by interchanging the rows and the columns of A.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -2 \\ 4 & 10 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 10 \\ -1 & -2 & 9 \end{bmatrix}$$

Multiplication:

Definition. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be matrices. Then $A\,B$ is the $m \times p$ matrix C, where

$$C = [c_{ij}]_{m\times p} = \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{m\times p}$$

Remark. $A\,B \neq B\,A$ necessarily.

Example:

$$\begin{bmatrix} 1 & -1 & 0 \\ 4 & 1 & -1 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 3 & 4 \\ -1 & -5 \\ 1 & 2 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} (1)(3)+(-1)(-1)+(0)(1) & (1)(4)+(-1)(-5)+(0)(2) \\ (4)(3)+(1)(-1)+(-1)(1) & (4)(4)+(1)(-5)+(-1)(2) \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 4 & 9 \\ 10 & 9 \end{bmatrix}_{2\times 2}$$

The following occur often for matrices.

1. $A\,B \neq B\,A$
2. $A\,B = 0$ but neither $A = 0$ or $B = 0$

3. $A\,B = A\,C$ but $B \neq C$

Theorem

Assume that k is an arbitrary scalar, and that $A,\ B,\ C$ and I are matrices of sizes such that the indicated operations can be performed. Then

1. $IA = A, \qquad BI = B$

2. $A(BC) = (AB)C$

3. $A(B+C) = AB+AC, \qquad A(B-C) = AB-AC$

4. $(B+C)A = BA+CA, \qquad (B-C)A = BA-CA$

5. $k(AB) = (kA)B = A(kB)$

6. $(AB)^T = B^TA^T$.

Cramer's Rule

Cramer's Rule: Let A be an $n \times n$ matrix, $A = [a_{ij}]_{n\times n}$ and denote by $A_{(j)}$ the $n \times n$ matrix formed by replacing the elements a_{ij} of the jth column of A by the numbers $k_i,\ i = 1,\ldots\ldots,n$. If $|A| \neq 0$, the system of n linear equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2 \\ \vdots &= \vdots \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= k_n \end{aligned}$$

has the unique solution

$$x_1 = \frac{\det A_{(1)}}{\det A} \qquad x_2 = \frac{\det A_{(2)}}{\det A}, \ldots x_n = \frac{\det A_{(n)}}{\det A}$$

Example. Solve

$$\begin{aligned} x + 3y - 2z &= 1 \\ 4x - 2y + z &= -15 \\ 3x + 4y - z &= 3 \end{aligned}$$

by Cramer’s Rule

$$\det A = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$$

$$x = \frac{\begin{vmatrix} 1 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = -\frac{14}{5}, \quad y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 4 & -15 & 1 \\ 3 & 3 & -1 \end{vmatrix}}{-25} = \frac{19}{5}, \quad z = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 4 & -1 & -15 \\ 3 & 4 & 3 \end{vmatrix}}{-25} = \frac{19}{5}$$

Systems of Equations: Elimination Using Matrices

Elementary Row Operations On Matrices I

Equivalent Systems

Two linear systems are **equivalent** if they have the same solutions.

Three Elementary Operations

Three basic elementary operations are used to transform systems to equivalent systems. These are:

1. Interchanging the order of the equations in the system.
2. Multiplying any equation by a nonzero constant.
3. Replacing any equation in the system by its sum with a nonzero constant multiple of any other equation in the system (elimination step).

Theorem:

Suppose that an elementary row operation is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

Operating on the rows of a matrix is equivalent to operating on equations. The row operations that are allowed are the same as the row operations on linear systems of equations:

1. Interchanging the rows.
2. Multiplying any row by a nonzero constant.
3. Replacing any row by its sum with a nonzero constant multiple of any other row. (Add a multiple of one row to a different row.)

Gaussian Elimination

Definition: A matrix is said to be in row-echelon form (and will be called a row-echelon matrix) if it satisfies the following three conditions:

1. All zero rows (consisting entirely of zeroes) are at the bottom.

2. The first nonzero entry from the left in each nonzero row is a $1,$ called the leading 1 for that row.

3. Each leading 1 is to the right of all leading $1's$ in the rows above it.

Definition: A row-echelon matrix is said to be in reduced row-echelon form (and will be called a reduced row-echelon matrix) if it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

Example:
Reduce the matrix

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix}$$

to row-reduced echelon form.

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix} \xrightarrow{(2)R_1+R_3;\ (2)R_1+R_4} \begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & -1 & 0 & 4 & -8 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{(-1)R_1;\ (-1)R_3} \begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix} \xrightarrow{(-1)R_3+R_4} \begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(-1)R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example:
Solve the system $AX = C,$ where

$$A = \begin{bmatrix} -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & -7 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} \text{ and } C = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 8 \end{bmatrix}$$

From the previous example

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solution is: $\{x_3 = 3x_4, x_1 = -2x_4 - 4, x_2 = 4x_4 + 8, x_4 = x_4\}$

Inverse of a Matrix

Definition: If A is a square $n \times n$ matrix, a matrix A^{-1} is called the inverse of A if and only if

$$AA^{-1} = I = A^{-1}A$$

A matrix A that has an inverse is called an invertible or nonsingular matrix.

Example:

Find A^{-1} for $A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. We form $\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \to_{(-2)R_2+R_1;\ (-1)R_2+R_3} \begin{bmatrix} 0 & -1 & 3 & 1 & -2 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\to_{(-1)R_1+R_3;} \begin{bmatrix} 0 & -1 & 3 & 1 & -2 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix}$$

$$\to_{(4)R_1+R_2;\ \left(-\frac{1}{2}\right)R_3} \begin{bmatrix} 0 & -1 & 3 & 1 & -2 & 0 \\ 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \to_{(-3)R_3+R_1;\ (-11)R_3+R_2} \begin{bmatrix} 0 & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\to_{(-1)R_1;\ R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Eigenvalues

Definition: The values of λ such that $\det(A - \lambda I) = 0$ are called eigenvalues. The vector X corresponding to an eigenvalue is called an eigenvector of the matrix A.

Example. Find all eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Solution:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \\ &= -2 + \lambda + 2\lambda^2 - \lambda^3 \\ &= -\lambda^2(\lambda - 2) + (\lambda - 2) \\ &= (1 - \lambda^2)(\lambda - 2) \end{aligned}$$

Thus $\det(A - \lambda I) = 0 \Rightarrow$eigenvalues $\lambda = -1, 1, 2.$
$(A - \lambda I)X = 0 \Rightarrow$

$$\begin{aligned} (1-\lambda)x_1 + x_2 - 2x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + x_2 + (-1-\lambda)x_3 &= 0 \end{aligned}$$

$\lambda = -1$

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 0 \\ -x_1 + 3x_2 + x_3 &= 0 \\ 0x_1 + x_2 + 0x_3 &= 0 \end{aligned}$$

Thus $x_2 = 0, x_1 = x_3$ or $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow -1.$ Similarly,

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \text{ eigenvectors: } \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\} \leftrightarrow 2, \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Matrix Methods for Linear Systems of Differential Equations

Linear Systems in Normal Form

A system of n linear differential equations is in normal form if it is expressed as

$$x'(t) = A(t)x(t) + f(t)$$

where $x(t)$ and $f(t)$ are $n \times 1$ column vectors and $A(t) = [a_{ij}(t)]_{n\times n}$.
A system is called homogeneous if $f(t) = 0;$ otherwise it is called nonhomogeneous. When the elements of A are constants, the system is said to have constant coefficients.

Example:
Express the equation

$$y''' - 6y'' + 11y' - 6ty = \cos t$$

in normal form

$$x'(t) = A(t)x(t) + f(t)$$

Solution: Defining

$$x_1 = y, x_2 = y', x_3 = y''$$

we have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 6tx_1 - 11x_2 + 6x_3 + \cos t \end{aligned}$$

Thus

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6t & -11 & 6 \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

Solving Normal Systems

1. To determine a general solution to the $n \times n$ homogeneous system $x' = Ax$:
  - **a**. Find a fundamental solution set $\{x_1, \ldots\ldots, x_n\}$ that consists of n linearly independent solutions to the homogeneous equation.
  - **b**. Form the linear combination

$$x = Xc = c_1x_1 + \cdots + c_nx_n$$

where $c = col(c_1, \ldots\ldots, c_n)$ is any constant vector and $X = [x_1, \ldots\ldots, x_n]$ is the fundamental matrix, to obtain a general solution.

Theorem

Suppose the $n \times n$ constant matrix A has n linearly independent eigenvectors $u_1, u_2, \ldots, u_n$. Let r_i be the eigenvalue corresponding to the u_i. Then

$$\{e^{r_1 t}u_1, e^{r_2 t}u_2, \ldots, e^{r_n t}u_n\}$$

is a fundamental solution set on $(-\infty, \infty)$ for the homogeneous system $x' = Ax$. Hence the general solution of $x' = Ax$ is

$$x(t) = c_1 e^{r_1 t}u_1 + \cdots + c_n e^{r_n t}u_n$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Example
Find a general solution of

$$x' = \begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix} x$$

$\begin{bmatrix} 5 & 4 \\ -1 & 0 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\} \leftrightarrow 4$

Thus $x(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$
Thus the solution is

$$x_1(t) = -c_1 e^t - 4c_2 e^{4t}$$
$$x_2(t) = c_1 e^t + c_2 e^{4t}$$

Example:
Find a fundamental matrix for the system

$$x'(t) = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} x(t)$$

Solution:

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \text{ eigenvectors: } \left\{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} -1 \\ 1 \\ 2 \\ 8 \end{bmatrix}\right\} \leftrightarrow 7,$$

$$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 3$$

Hence the four linearly independent solutions are

$$e^{-t}\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}, e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e^{7t}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 8 \end{bmatrix}, e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore a fundamental matrix is

$$\begin{bmatrix} e^{-t} & e^{2t} & -e^{7t} & e^{3t} \\ -3e^{-t} & 0 & e^{7t} & 0 \\ 0 & 0 & 2e^{7t} & e^{3t} \\ 0 & 0 & 8e^{7t} & 0 \end{bmatrix}$$

Example:

Solve the initial value problem

$$x'(t) = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} x(t)$$

$$x(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

We know from above that the solution general solution to the system is

$$x(t) = c_1e^{-t}\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} + c_2e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3e^{7t}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 8 \end{bmatrix} + c_4e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} c_1e^{-t} + c_2e^{2t} - c_3e^{7t} + c_4e^{3t} \\ -3c_1e^{-t} + c_3e^{7t} \\ 2c_3e^{7t} + c_4e^{3t} \\ 8c_3e^{7t} \end{bmatrix}$$

Then $x(0) = \begin{bmatrix} c_1 + c_2 - c_3 + c_4 \\ -3c_1 + c_3 \\ 2c_3 + c_4 \\ 8c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$. Therefore we must solve the system

$$\begin{aligned} c_1 + c_2 - c_3 + c_4 &= 1 \\ -3c_1 + c_3 &= -1 \\ 2c_3 + c_4 &= 1 \\ 8c_3 &= 0 \end{aligned}$$

, Solution is: $\{c_3 = 0, c_4 = 1, c_1 = \frac{1}{3}, c_2 = -\frac{1}{3}\}$,and

$$x(t) = \begin{bmatrix} \frac{1}{3}e^{-t} - \frac{1}{3}e^{2t} + e^{3t} \\ -e^{-t} \\ e^{3t} \\ 0 \end{bmatrix}$$

Complex Eigenvalues

Consider

$$x'(t) = Ax(t) \qquad (*)$$

in the case where A is a real matrix and the eigenvalues are complex. Denoting the eigenvalues by $\alpha \pm i\beta$, let $\mathbf{z} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real vectors, be an eigenvector corresponding to the eigenvector r_1. Then

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t}(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b}) \\ \mathbf{x}_2(t) &= e^{\alpha t}(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b}) \end{aligned}$$

are two real lineraly independent solutions of the system $(*)$.

Find the general solution of

$$x'(t) = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} x(t)$$

Solution: This is problem 1 on page 573 of our DEs text and was assigned for homework.

Eigenvalues:

$$\det(A - rI) = \begin{vmatrix} 2-r & -4 \\ 2 & -2-r \end{vmatrix} = r^2 + 4 = 0 \Rightarrow r = \pm 2i = \alpha \pm i\beta, \text{ so } \alpha = 0, \beta = 2$$

Eigenvectors:

$r = 2i$:

$$\begin{bmatrix} 2-2i & -4 \\ 2 & -2-2i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2-2i & -4 & 0 \\ 2 & -2-2i & 0 \end{bmatrix}$$

R_1 says $(2-2i)u_1 = 4u_2 \Rightarrow u_2 = \left(\frac{2-2i}{4}\right)u_1 = \left(\frac{1}{2} - \frac{i}{2}\right)u_1$. Let $u_1 = s$;

then $\vec{u} = \begin{bmatrix} s \\ \left(\frac{1}{2} - \frac{i}{2}\right)s \end{bmatrix} = s\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + is\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$. Let $s = 2$:

$\Rightarrow$

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \vec{a} + i\vec{b}.$$

So the general solution is

$$\begin{aligned} \vec{x}(t) &= c_1\left\{e^{0t}\cos 2t\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{0t}\sin 2t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\} + c_2\left\{e^{0t}\sin 2t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{0t}\cos 2t\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\} \\ &= c_1\begin{bmatrix} 2\cos 2t \\ \cos 2t + \sin t2t \end{bmatrix} + c_2\begin{bmatrix} 2\sin 2t \\ \sin 2t - \cos 2t \end{bmatrix} \end{aligned}$$

Nonhomogeneous Systems

Undetermined Coefficients

Consider the nonhomogeneous constant coefficient system

$$x'(t) = Ax(t) + f(t)$$

Find the general solution of

$$x'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}x(t) + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$$

Solution:

We first find the homogeneous solution.

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \text{ eigenvectors: } \left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow -3, \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \leftrightarrow 3$$

Since these eigenvectors are linearly independent, then

$$x_h(t) = c_1e^{-3t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We seek a particular solution of the form

$$x_p(t) = e^t\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Then

$$x_p'(t) = e^t\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = Ax_p(t) = e^t\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$$

$$= e^t\left(\begin{bmatrix} a_1 - 2a_2 + 2a_3 \\ -2a_1 + a_2 + 2a_3 \\ 2a_1 + 2a_2 + a_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}\right)$$

Thus

$$\begin{aligned} a_1 &= a_1 - 2a_2 + 2a_3 + 2 \\ a_2 &= -2a_1 + a_2 + 2a_3 + 4 \\ a_3 &= 2a_1 + 2a_2 + a_3 - 2 \end{aligned}$$

, Solution is: $\{a_2 = 0, a_1 = 1, a_3 = -1\}$

Therefore

$$x_p(t) = e^t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$x(t) = x_h(t) + x_p(t) = c_1e^{-3t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + e^t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Note: the Method of Undetermined Coefficients works only for constant coefficient systems.

The Matrix Exponential Function

Definition: Let A be a constant $n \times n$ matrix. The we define

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} A^n\frac{t^n}{n!}$$

This is an $n \times n$ matrix.

Remark: If D is a diagonal matrix, then the computation of e^{At} is straightforward.

Example

Let $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad D^3 = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}, \ldots. \qquad D^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

Therefore

$$e^{Dt} = \sum_{n=0}^{\infty} D^n\frac{t^n}{n!} = \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} 2^n\frac{t^n}{n!} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

In general if D is an $n \times n$ diagonal matrix with $r_1, r_2, \ldots, r_n$ down its main diagonal, then e^{Dt} is the diagonal matrix with $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$ down its main diagonal.

In general it is not easy to calculate e^{At} for any matrix A.

Remark: In one case it is relatively easy to find e^{At}. It can be shown that if a matrix A has n linearly independent eigenvectors, then $P^{-1}AP$ is a diagonal matrix, where P is formed from the n linearly independent eigenvectors of A. Thus

$$P^{-1}AP = D \tag{$*$}$$

where D is a diagonal matrix. In fact, D has the eigenvalues of A along its diagonal. Now $(*)$ implies that when A has n linearly independent eigenvalues we have

$$A = PDP^{-1}$$

so that

$$\begin{aligned} e^{At} &= e^{PDP^{-1}t} = I + PDP^{-1}t + \frac{1}{2}(PDP^{-1}t)(PDP^{-1}t) + \cdots \\ &= I + PDP^{-1}t + \frac{1}{2}(PDP^{-1})(PDP^{-1})t^2 + \cdots \\ &= I + PDP^{-1}t + \frac{1}{2}(PD^2P^{-1})t^2 + \cdots \\ &= P\left(I + Dt + \frac{1}{2}(Dt)^2 + \cdots\right)P^{-1} \\ &= Pe^{Dt}P^{-1} \end{aligned}$$

Example:

We saw above that $A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$, has eigenvectors:

$$\left\{\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} -1 \\ 1 \\ 2 \\ 8 \end{bmatrix}\right\} \leftrightarrow 7, \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 3$$

Let

$$P = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 8 & 0 \end{bmatrix}$$

Then $P^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 & \frac{1}{24} \\ 1 & \frac{1}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 1 & -\frac{1}{4} \end{bmatrix}$ so

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = D$$

Thus

$$e^{Dt} = e^{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} t}$$

$$e^{Dt} = \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{7t} & 0 \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

Hence

$$e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{7t} & 0 \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}P^{-1}$$

$$= \begin{bmatrix} e^{2t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} & -e^{2t} + e^{3t} & \frac{1}{24}e^{-t} + \frac{1}{3}e^{2t} - \frac{1}{8}e^{7t} - \frac{1}{4}e^{3t} \\ 0 & e^{-t} & 0 & -\frac{1}{8}e^{-t} + \frac{1}{8}e^{7t} \\ 0 & 0 & e^{3t} & \frac{1}{4}e^{7t} - \frac{1}{4}e^{3t} \\ 0 & 0 & 0 & e^{7t} \end{bmatrix}$$

Calculating e^{At} for Nilpotent Matrices

Definition: An *nxn* matrix *A* matrix is nilpotent if for some positive integer *k*

$$A^k = 0.$$

Since

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} A^n\frac{t^n}{n!}$$

we see that if A is nilpotent, then the infinite series has only a finite number of terms since $A^k = A^{k+1} = \cdots = 0$ and in this case

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^{k-1}\frac{t^{k-1}}{(k-1)!}$$

This may be taken further. The Cayley-Hamilton Theorem says that a matrix satisfies its own characteristic equation, that is, $p(A) = 0.$ Therefore, if the characteristic polynomial for A has the form $p(r) = (-1)^n(r - r_1)^n,$ that is A has only one multiple eigenvalue $r_1,$ then $p(A) = (-1)^n(A - r_1 I)^n = 0.$ Hence $A - r_1 I$ is nilpotent and

$$e^{At} = e^{(r_1 I + A - r_1 I)t} = e^{r_1 I t}e^{(A - r_1 I)t} = e^{r_1 t}\left[I + (A - r_1 I)t + \cdots + (A - r_1 I)^{n-1}\frac{t^{n-1}}{(n-1)!}\right]$$

Example Find the fundamental matrix e^{At} for the system

$$x'(t) = Ax(t) \text{ where } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$$

Solution: The characteristic polynomial for A is

$$p(r) = \det\begin{bmatrix} 2-r & 1 & 1 \\ 1 & 2-r & 1 \\ -2 & -2 & -1-r \end{bmatrix} = -r^3 + 3r^2 - 3r + 1 = -(r-1)^3$$

Hence $r = 1$ is an eigenvalue of A with multiplicity 3. By the Cayley-Hamilton Theorem $(A-I)^3 = 0$ and

$$e^{At} = e^t e^{(A-I)t} = e^t\left[I + (A-I)t + (A-I)^2\frac{t^2}{2!}\right]$$

$$A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \text{ and } (A-I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$e^{At} = e^t\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^t\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} e^t + te^t & te^t & te^t \\ te^t & e^t + te^t & te^t \\ -2te^t & -2te^t & e^t - 2e^t \end{bmatrix}$$