ABSTRACT

Nth ORDER DIFFERENTIAL EQUATIONS WITH FINITE POLYNOMIAL SOLUTIONS

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Drawing on previous results which give conditions the existence of two linearly independent polynomial solutions for the equation

$$(1-x^2)y'' + Axy' + By = 0, \qquad (*)$$

where $y = y(x)$ and A and B are constants, the authors consider the equations

$$(1-x^3)y''' + Ax^2y'' + Bxy' + Cy = 0 \qquad (**)$$

and

$$(1-x^4)y^{(iv)} + Ax^3y''' + Bx^2y'' + Cxy' + Dy = 0. \qquad (***)$$

Conditions are given on $A, B,$ and C that guarantee that $(**)$ has three linearly independent polynomial solutions. Similar results are obtained for the constants in $(***)$ so that this equation has four linearly independent polynomial solutions. The degrees of these solutions are also dealt with.

Continuing in this manner, the authors conclude with remarks about the N-th order generalization of the above equations and indicate analogous results.

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1. Introduction

In [1] it was shown that an equation of the form

(1) $$(1-x^2)y'' + Axy' + By = 0$$

has two linearly independent polynomial solutions if and only if

(2) $$A = a + b - 1$$

and

(3) $$B = -ab$$

where a and b are positive integers and $a + b$ is odd. Furthermore, these solutions have degrees a and b.

2. THE THIRD ORDER CASE

Consider the equation

(4) $$(1-x^3)y''' + Ax^2y'' + Bxy' + Cy = 0$$

where $A, B,$ and C are constants. Then the following results hold.

Theorem 1: Equation (4) has three finite polynomial solutions if and only if

(5) $$A = a + b + c - 3,$$

(6) $$B = a + b + c - ab - ac - bc - 1,$$

and

(7) $$C = abc$$

where $a, b,$ and c are positive integers and $a \equiv 0 \bmod 3$, $b \equiv 1 \bmod 3$, and $c \equiv 2 \bmod 3$.
Moreover, the existing polynomial solutions will be of degree $a, b,$ and c .

Proof. Observe that $x = 0$ is an ordinary point of (4). Assume that $y = \sum a_n x^n$. Substitution into (4), followed by the usual manipulation of indices, gives the following formula:

(8) $$a_{n+3} = \frac{f(n;A,B,C)}{[(n+1)(n+2)(n+3)]} a_n, \qquad \text{for all } n$$

where $f(n;A,B,C)$ is some function of n and the parameters $A, B,$ and C. Because $a_0, a_1,$ and a are arbitrary, three finites polynomial solutions will be assured if and only if $f(n;A,B,C)$ has the form $(n-a)(n-b)(n-c)$ where $a, b,$ and c are positive integers and $a \equiv 0 \bmod 3$, $b \equiv 1 \bmod 3$, and $c \equiv 2 \bmod 3$. Rewriting (8) as

(9) $$a_{n+3} = \frac{(n-a)(n-b)(n-c)}{[(n+1)(n+2)(n+3)]} a_n$$

leads to the following relation

(10) $$(n+1)(n+2)(n+3)a_{n+3} - (n-a)(n-b)(n-c)a_n = 0$$

where $n = 0, 1, 2, \ldots$ Multiplying both sides of (10) by x^n, and summing over all n, brings us to the equation

(11) $$(1-x^3)y''' + (a+b+c-3)y'' + (a+b+c-ab-ac-bc-1)xy' + abcy = 0$$

which completes our proof.

Example 1: The equation

(12) $$(1-x^3)y''' + 3x^2y'' - 6xy' + 6y = 0$$

has three finite polynomial solutions: $y_1 = 1 - x^3, y_2 = x,$ and $y_3 = x^2$. Here $A = 3, B = -6, C = 6$ with $a = 3,\ b = 1$ and $c = 2$.
Example 2: The equation

(13) $$(1 - x^3)y''' + 4x^2y'' - 7xy' + 8y = 0$$

does not possess three finite polynomial solutions for the system of equations

(14) $$\left.\begin{aligned} A &= 4 = a + b + c - 3 \\ B &= -7 = a + b + c - ab - ac - bc - 1 \\ C &= 8 = abc \end{aligned}\right\}$$

does not possess the required integral solutions.

3. THE FOURTH ORDER CASE

By parallelling the technique in the previous section we can easily prove the following.
Theorem 2: Let A, B, C and D be constants. Then the equation

(15) $$(1 - x^4)y^{(iv)} + Ax^3y''' + Bx^2y'' + Cxy' + Dy = 0$$

has four finite polynomial solutions, if and only if

(16) $$A = a + b + c + d - 6,$$

(17) $$B = 3(a + b + c + d) - ab - ac - ad - bc - bd - cd - 7$$

(18) $$C = a + b + c + d - ab - ac - ad - bc - bd - cd + abc + abd + acd + bcd - 1$$

and

(19) $$D = -abcd$$

where $a, b, c,$ and d are positive integers such that $a \equiv 0 \bmod 4, b \equiv 1 \bmod 4, c \equiv 2 \bmod 4,$ and $d \equiv 3 \bmod 4$. These polynomials will have degree $a, b, c,$ and d .

4. CONCLUDING REMARKS

While euqations such as (4) and (15) do not appear as much in scientific literature as their first and second order counterparts do, instructors can use them while teaching power series techniques with regard to the verification of solutions to equations of order three and higher.
Finally, a note on the generalized equation.
Consider

(20) $$(1 - x^N)y^{(n)} + A_{N-1}x^{N-1}y^{(N-1)} + A_{N-2}x^{N-2}y^{(N-2)} + \cdots + A_1xy' + A_0y = 0$$

where , $A_m,\ m = 0, 1, \ldots, N-1$ are constants and $N > 4$. Here, N finite polynomial solutions, of degree $k,$ exist if and only if $k_s,\ s = 1, 2, \ldots, N$ are positive integers with

(21) $$k_t \equiv (t-1) \bmod N$$

where $t = 1, 2, \ldots, N$ and the system

(22) $$A_j = F_j(k_1, k_2, \ldots, k_N)$$

for $j = 0, 1, \ldots, N-1$ is appropriately satisfied.

5. REFERENCES

[1] G.B.Costa and L.E.Levine, 1996, *Classes of Second Order Differential Equations with Two Linearly Independent Finite Polynomial Solutions*, INTERNATIONALJOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE ANDTECHNOLOGY, Volume 27, 623-625, (Leicestershire, United Kingdom).
[2] STANDARD MATHEMATICAL TABLES (27th ed.), 1984, W.H. Beyer, Ed. CRC Press, Inc., (Boca Raton, Fla.)