Polynomial Solutions of the Classical Equations of Hermite, Legendre, and Chebyshev

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Abstract:

The classical differential equations of Hermite, Legendre, and Chebyshev are well-known for their polynomial solutions. These polynomials occur in the solutions to numerous problems in applied mathematics, physics, and engineering. However, since these equations are of second order, they also have second linearly independent solutions that are not polynomials. These solutions usually cannot be expressed in terms of elementary functions alone.

In this paper, the classical differential equations of Hermite, Legendre, and Chebyshev are studied when they have a forcing term x^M on the right hand side. It will be shown that for each equation, choosing a certain initial condition is a necessary and sufficient condition for ensuring a polynomial solution. Once this initial condition is determined, the exact form of the polynomial solution is presented.

Introduction:

The equations of Hermite, Legendre, and Chebyshev with a forcing term are all special cases of the equation

$$(\theta + \beta x^2)y'' + \gamma x y' + F(\ell)y = R(x) \tag{1}$$

where $\theta \neq 0$, and $\ell \geq 0$ is a parameter. Let $R(x) = x^M$. In the case of the three special equations, if ℓ is odd, then choose M to be even. Similarly, if ℓ is even, then choose M to be odd. The reason for this is as follows:

The recurrence relationship for (1) yields the auxiliary equation

$$\beta k^2 + (\gamma - \beta)k + F(\ell) = 0 \tag{2}$$

The recurrence relationship also implies the fact that each coefficient in the series expansion of the solution depends entirely on the coefficient two terms before it. It follows that one solution of (1) depends entirely on the initial condition $y(0)$ and that the other solution depends entirely on the initial condition $y'(0)$. Equation (2) has two (not necessarily distinct) solutions if $\beta \neq 0$ (which is the case for Legendre's and Chebyshev's equations) or one solution if $\beta = 0$ (which is the case for Hermite's equation). In each of these cases, one solution is the parameter ℓ. This implies that the recurrence relationship will yield a value of zero when the ℓth term is used to calculate the $(\ell + 2)$th term. Thus, for example, if ℓ is even, then the solution that depends on $y(0)$ is a polynomial. However, the solution that depends on $y'(0)$ is not a polynomial. This is why we choose a forcing term of x^M such that M and ℓ have different parity.

There are six different cases to consider. We now proceed to derive the necessary and sufficient initial conditions to ensure that these nonhomogeneous classical equations will possess only polynomial solutions. Then, we derive the exact forms of these polynomial solutions. For each equation, the required initial conditions, the fact the the solution is a polynomial, and the exact form of the polynomial solution will be shown to be equivalent statements.

At this point, we make note of a convention used throughout this paper. If a summation takes the form

$$\sum_{j=a}^{b} f(j)$$

where $a > b$, it is to be understood that the set of all terms $f(j)$ that will be summed is empty. In other words, the summation should be treated as if it did not exist at all.

Hermite Equation With An Even Parameter

Theorem 1:

Consider the initial value problem for the nonhomogeneous Hermite equation with the parameter $\ell = 2m$.

$$y'' - 2xy' + 2(2m)y = x^{2n+1}$$

$$y(0) = a$$

$$y'(0) = b$$

Then, the following statements are equivalent:

$$b = -(2n+1)! \prod_{i=0}^{n} \frac{1}{4i + 2 - 4m}$$

$$y(x) \text{ is a polynomial}$$

$$y(x) = a\frac{H_{2m}(x)}{H_{2m}(0)} - (2n+1)! \prod_{i=0}^{n} \left(\frac{1}{4i+2-4m} \right) x + \sum_{j=0}^{n-1} \left(\frac{-(2n+1)!}{(2j+3)!} \prod_{i=j+1}^{n} \left(\frac{1}{4i+2-4m} \right) x^{2j+3} \right)$$

where $H_{2m}(x)$ is the Hermite Polynomial of order $2m$

Proof:

Call the three statements above $A, B,$ and C respectively. We will prove the implications in this order: $A \rightarrow B, B \rightarrow A, B \rightarrow C,$ and finally $C \rightarrow B$. By transitivity, the equivalencies follow.

To show that $A \rightarrow B$, consider the substitution

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

which yields

$$\sum_{k=0}^{2n}\left([a_{k+2}(k+2)(k+1)-2a_k(k-2m)]x^k\right)+(a_{2n+3}(2n+3)(2n+2)-2a_{2n+1}(2n+1-2m)-1)x^{2n+1}$$

$$+\sum_{k=2n+2}^{\infty}([a_{k+2}(k+2)(k+1)-2a_k(k-2m)]x^k)=0$$

The coefficients a_0 to a_{2n+2} will be exactly the same as those in the homogeneous solution. Once the term a_{2n+3} is reached, the nonhomogeneous term takes effect. At this point, that term and all the odd terms thereafter can be eliminated with an appropriate choice of initial conditions.

To see this, we calculate a few terms of the homogenous solution. From our initial conditions we know that $a_0 = a$ and $a_1 = b$.

The recurrence relationship is:

$$a_{k+2}=\frac{2a_k(k-2m)}{(k+2)(k+1)}$$

so that

$$k=0\Rightarrow a_2=\frac{2a_0(-2m)}{(2)(1)}=-2am$$

$$k=1\Rightarrow a_3=\frac{2a_1(1-2m)}{(3)(2)}=\frac{1}{3}b(1-2m),$$

etc.

For the special case where $k = 2m$

$$a_{2m+2}=\frac{2a_{2m}(2m-2m)}{(2m+2)(2m+1)}=0.$$

This is a very important result. Now we know that a_{2m+2}, as well as all even terms thereafter, will be zero. Since these even terms cancel automatically, it is sufficient to only consider the initial condition $y'(0) = b$ when trying to reduce the complete solution to a finite polynomial. It should also be noted that the non-homogenous term has no effect on this property, since it only affects odd terms in the solution.

The next step is to generate a closed form formula for a_{k+2} when $k < 2n+1$ is odd.

$$a_{k+2}=\frac{2a_k(k-2m)}{(k+2)(k+1)}=\frac{(2)(k-2m)}{(k+2)(k+1)}\frac{(2)(a_{k-2})(k-2-2m)}{(k)(k-1)}=\cdots$$
$$=\frac{(2)(k-2m)(2)(k-2-2m)\cdots(2)(1-2m)b}{(k+2)!}$$

Thus,

$$a_{k+2}=\frac{(2k-4m)(2k-4-4m)\cdots(2-4m)b}{(k+2)!}$$

This will be particularly useful when determining a_{2n+3}.

$$a_{2n+3}=\frac{2a_{2n+1}(2n+1-2m)}{(2n+3)(2n+2)}+\frac{1}{(2n+3)(2n+2)}$$

Notice how $\frac{2a_{2n+1}(2n+1-2m)}{(2n+3)(2n+2)}$ would be the value of a_{2n+3} in the homogeneous case.

Thus,

$$a_{2n+3} = \frac{(4n+2-4m)(4n-2-4m)\cdots(2-4m)b}{(2n+3)!} + \frac{1}{(2n+3)(2n+2)}$$

Now we set a_{2n+3} to zero. If this term becomes zero, then all odd terms thereafter must follow. And since we have also shown that beyond a certain point all even terms will be zero, then the complete solution must be a finite polynomial. Therefore, we want

$$b = \frac{-(2n+1)!}{(4n+2-4m)(4n-2-4m)\cdots(2-4m)}.$$

or

$$b = -(2n+1)!\prod_{i=0}^{n}\frac{1}{4i+2-4m} \tag{3}$$

Choosing this initial condition implies that the initial value problem will have a polynomial solution, establishing the first result.

To show the $B \to A$, consider the auxiliary equation for Hermite's equation with an even parameter, which is:

$$-2k + 2\ell = 0$$

This yields the solution

$$k = \ell = 2m$$

which is as expected. Now we show the contrapositive $\sim A \to \sim B$. Suppose that the value of b is different than that shown in (3). Then, $a_{2n+3} \neq 0$. But then:

$$a_{2n+5} = \frac{2a_{2n+3}(2n+3-2m)}{(2n+5)(2n+4)} \neq 0$$

since $a_{2n+3} \neq 0$ and $2n+3$ is not a solution to the auxiliary equation, which implies that $2n+3-2m \neq 0$. Also, $a_{2n+(2s+1)} \neq 0$ and $2n+(2s+1) \neq 2m$ implies that $a_{2n+(2s+3)} \neq 0$. Thus, by induction, $y(x)$ is not a polynomial, establishing the result.

We now show that $B \to C$. Since we are dealing with a Hermite equation with an even parameter and assuming a polynomial solution, the solution will take the form:

$$y(x) = cH_{2m}(x) + \sum_{j=0}^{n} a_{2j+1}x^{2j+1}$$

where c is an arbitrary constant and H_{2m} is the Hermite polynomial of order $2m$. The order of the second solution is known to be $2n+1$ since $a_{2n+3} = 0$. Rewrite this as:

$$y(x) = cH_{2m}(x) + a_1 x + \sum_{j=0}^{n-1} a_{2j+3}x^{2j+3}$$

Now substitute the appropriate values for the various a_i's:

$$y(x) = cH_{2m}(x) + bx + \sum_{j=0}^{n-1}\frac{(4j+2-4m)(4j-2-4m)\cdots(2-4m)}{(2j+3)!}bx^{2j+3}$$

The next step is to substitute the appropriate value of b (since $B \to A$) into the expression:

$$y(x) = cH_{2m}(x) - (2n+1)!\prod_{i=0}^{n}\frac{1}{4i+2-4m}x$$
$$+\sum_{j=0}^{n-1}\left(\frac{(4j+2-4m)(4j-2-4m)\cdots(2-4m)}{(2j+3)!}\left(-(2n+1)!\prod_{i=0}^{n}\frac{1}{4i+2-4m}\right)x^{2j+3}\right)$$

This reduces to:

$$y(x) = cH_{2m}(x) - (2n+1)!\prod_{i=0}^{n}\frac{1}{4i+2-4m}x + \sum_{j=0}^{n-1}\left(\frac{-(2n+1)!}{(2j+3)!}\left(\prod_{i=j+1}^{n}\frac{1}{4i+2-4m}\right)x^{2j+3}\right)$$

To determine the value of c, let's examine $y(0)$. This is simply $H_{2m}(0)$. This gives us $a = cH_{2m}(0)$. Thus,

$$y(x) = a\frac{H_{2m}(x)}{H_{2m}(0)} - (2n+1)!\prod_{i=0}^{n}\left(\frac{1}{4i+2-4m}\right)x + \sum_{j=0}^{n-1}\left(\frac{-(2n+1)!}{(2j+3)!}\prod_{i=j+1}^{n}\left(\frac{1}{4i+2-4m}\right)x^{2j+3}\right)$$

where $H_{2m}(x)$ is the Hermite Polynomial of order $2m$.

It remains to show that $C \to B$. This implication is actually quite obvious. Statement C clearly is a polynomial. Having shown these implications, it follows that statements $A, B,$ and C are all equivalent.

□

Hermite Equation With An Odd Parameter

Theorem 2:

Given the initial value problem

$$y'' - 2xy' + 2(2m+1)y = x^{2n}$$
$$y(0) = a$$
$$y'(0) = b$$

Then, the following statements are equivalent:

$$a = -(2n)!\prod_{i=0}^{n}\left[\frac{1}{4i-4m-2}\right]$$

$y(x)$ is a polynomial

$$y(x) = b\frac{H_{2m+1}(x)}{\frac{d}{dx}H_{2m+1}(x)|_{x=0}} - (2n)!\prod_{i=0}^{n}\left(\frac{1}{4i-4m-2}\right) + \sum_{j=0}^{n-1}\left(\frac{-(2n)!}{(2j+2)!}\prod_{i=j+1}^{n}\left(\frac{1}{4i-4m-2}\right)x^{2j+2}\right)$$

where $H_{2m+1}(x)$ is the Hermite polynomial of order $2m+1$.

Proof:

The proof of this theorem is very similar to that of Theorem 1.

□

Legendre Equation With An Even Parameter

Theorem 3:

Given the initial value problem with the parameter $\ell = 2m$

$$(1-x^2)y'' - 2xy' + (2m)(2m+1)y = x^{2n+1}$$
$$y(0) = a$$
$$y'(0) = b$$

Then, the following statements are equivalent

$$a = -(2n+1)!\prod_{i=0}^{n} \frac{1}{(2i+1)(2i+2)-(2m)(2m+1)}$$

$$y(x) \text{ is a polynomial}$$

$$y(x) = a\frac{P_{2m}(x)}{P_{2m}(0)} - (2n+1)!\prod_{i=0}^{n}\left(\frac{1}{(2i+1)(2i+2)-(2m)(2m+1)}\right)x + \sum_{j=0}^{n-1}\left(\frac{-(2n+1)!}{(2j+3)!}\left(\prod_{i=j+1}^{n}\frac{1}{(2i+1)(2i+2)-(2m)(2m+1)}\right)x^{2j+3}\right)$$

where $P_{2m}(x)$ is the Legendre polynomial of order $2m$.

Proof:

The proof of this theorem is exactly the same as that for Theorems 1-2; however, there is a small note that should be mentioned: Recall that the auxiliary equation for the Hermite equation has only one root while the auxiliary equations for the Legendre and Chebyshev equations each have two roots. As a result, we must examine the auxiliary equations more carefully when working with these differential equations. The auxiliary equation for Legendre's equation is:

$$-k^2 - k + (\ell)(\ell+1) = 0$$

which has the solutions

$$k = \{-\ell-1, \ell\}$$

Regardless of the parity of ℓ, one solution is greater than or equal to zero, while the other is always negative (since $\ell \geq 0$). The negative solution can be ignored since the indices of the coefficients in the series expansion of the solution are all greater than or equal to zero. Thus, the method of proof used for Theorem 1 is still valid.

□

Legendre Equation With An Odd Parameter

Theorem 4:

Given the initial value problem with the parameter $\ell = 2m-1$

$$(1-x^2)y'' - 2xy' + (2m-1)(2m)y = x^{2n}$$
$$y(0) = a$$
$$y'(0) = b$$

Then, the following statements are equivalent:

$$a = -(2n)!\prod_{i=0}^{n} \frac{1}{(2i)(2i+1)-(2m)(2m-1)}$$

$$y(x) \text{ is a polynomial}$$

$$y(x) = b\frac{P_{2m-1}(x)}{P'_{2m-1}(0)} - (2n)!\prod_{i=0}^{n}\left(\frac{1}{(2i)(2i+1)-(2m)(2m-1)}\right)$$
$$+\sum_{j=0}^{n-1}\left(\frac{-(2n)!}{(2j+2)!}\left(\prod_{i=j+1}^{n}\frac{1}{(2i)(2i+1)-(2m-1)(2m)}\right)x^{2j+2}\right)$$

where $P_{2m-1}(x)$ is the Legendre Polynomial of order $2m-1$.
Proof:
The proof of this theorem parallels the proofs of Theorems 1-3.
□

Chebyshev Equation With An Even Parameter

Theorem 5:
Given the initial value problem with the parameter $\ell = 2m$

$$(1-x^2)y'' - xy' + (2m)^2 y = x^{2n+1}$$
$$y(0) = a$$
$$y'(0) = b$$

Then, the following statements are equivalent:

$$b = -(2n+1)!\prod_{i=0}^{n}\frac{1}{((2i+1)^2-4m^2)}$$

$$y(x) \text{ is a polynomial}$$

$$y(x) = a\frac{T_{2m}(x)}{T_{2m}(0)} - (2n+1)!\prod_{i=0}^{n}\frac{1}{((2i+1)^2-4m^2)} + \sum_{j=0}^{n-1}(\frac{-(2n+1)!}{(2j+3)!}(\prod_{i=j+1}^{n}\frac{1}{((2i+1)^2-4m^2)})x^{2j+3})$$

where $T_{2m}(x)$ is the Chebyshev polynomial of order $2m$.
Proof:
This proof is once again very similar to the previous ones. However, we must check the auxiliary equation just to make sure that the method of proof is valid in this case. This equation is:

$$-k^2 + \ell^2 = 0$$

which has the solution

$$k = \pm\ell$$

In the case where $\ell > 0$, one solution is positive and the other one is negative. The negative one can be ignored since the indices of the coefficients in the solution are greater than or equal to zero. Thus, if $\ell > 0$, we are safe using the same method as before. In the case $\ell = 0$, 0 becomes a repeated root. In this case, 0 acts as if it was a single root. Thus, the method of proof is also valid in this case as well.
□

Chebyshev Equation With An Odd Parameter

Theorem 6:
Given the initial value problem with the parameter $\ell = 2m + 1$

$$(1 - x^2)y'' - xy' + (2m + 1)^2 y = x^{2n}$$
$$y(0) = a$$
$$y'(0) = b$$

Then, the following statements are equivalent:

$$a = -(2n)! \prod_{i=0}^{n} \frac{1}{4i^2 - (2m + 1)^2}$$

$y(x)$ is a polynomial

$$y(x) = b \frac{T_{2m+1}(x)}{\frac{d}{dx} T_{2m+1}(x)|_{x=0}} - (2n)! \prod_{i=0}^{n} \left(\frac{1}{4i^2 - (2m + 1)^2} \right) + \sum_{k=0}^{n-1} \left(\frac{-(2n)!}{(2k + 2)!} \prod_{i=k+1}^{n} \left(\frac{1}{4i^2 - (2m + 1)^2} \right) x^{2k+2} \right)$$

where $T_{2m+1}(x)$ is the Chebyshev polynomial of order $2m + 1$.
Proof:
This proof is exactly the same as before.
□

Concluding Remarks

By modifying the classical equations of Hermite, Legendre, and Chebyshev so that they include the forcing term x^M, we were able to force their solutions into becoming polynomials. Associated with the nonhomogeneous terms are appropriate initial conditions that had to be satisfied. It was shown that polynomial solutions exist if and only if these initial conditions are satisfied. Since there is no other way for the solutions to become polynomials, the initial conditions are necessary and sufficient for the forms of the solutions.

The results presented also can be applied to more general equations of higher order. For example, consider the equation

$$(\theta + A_N x^N) y^{(N)} + A_{N-1} x^{N-1} y^{(N-1)} + A_{N-2} x^{N-2} y^{(N-2)} + \cdots + A_1 x y' + A_0 y = 0$$

Choosing a forcing term x^M will truncate a homogeneous solution into a polynomial by setting the arbitrary constant in front of the homogeneous solution to zero and creating a particular solution that matches a finite portion of the homogeneous solution. Of course, this is all provided that the homogeneous solution is not already a polynomial. Proceeding with calculations similar to those presented above yields the initial condition

$$y^{(r)}(0) = -\theta^q (qN + r)! \prod_{i=0}^{q} \left(-A_0 - \sum_{j=1}^{N} \left(A_j \prod_{l=0}^{j-1} (M - iN - l) \right) \right)^{-1} \tag{4}$$

where $M = qN + r$.

If the equation has $1 < s \leq N$ infinite series solutions, then s different values of M must be chosen such that each M belongs to a different congruence class modulo N and if for some M, $M \in \bar{r}$, then $y^{(r)}(0)$ applies to one of the infinite series homogeneous solutions. Then, by the superposition principle, the forcing term would be the sum of these s x^M's. Furthermore, the s initial conditions must be chosen by using (4). This would ensure a polynomial solution.

Now that the complete solution of any of the classical equations can be forced to become a polynomial, the entire solution should be of interest to mathematicians. The next step is to study the resulting particular solutions. For example, since the naturally occurring polynomial solutions of the classical equations enjoy orthogonality conditions, there is the question of whether these forced polynomials share a similar property.