Polynomial Solutions of Nth Order Nonhomogeneous Differential Equations

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Abstract:

It was shown in [1] that the homogeneous differential equation

$$(1-x^N)y^{(N)} + A_{N-1}x^{N-1}y^{(N-1)} + A_{N-2}x^{N-2}y^{(N-2)} + \cdots + A_1xy' + A_0y = 0 \tag{1}$$

has a finite polynomial solution if and only if

$$\forall r \ni 0 \le r < N, \exists n \ge 0 \ni n \bmod N = r \tag{2}$$

where n is a root of the recurrence relation.

In this paper, the case in which the equation has a forcing term on the right hand side is considered. This forcing term is selected in such a manner that, given appropriate initial conditions, a particular solution will result that matches a finite portion of the infinite series homogenous solution, and at the same time, annihilates this infinite series homogeneous solution. The result of these initial conditions and this right hand side is a solution that is a polynomial.

The results obtained apply to the classical equations of Hermite, Legendre, and Chebyshev with appropriate forcing terms and initial conditions.

Introduction:

Consider the initial value problem:

$$\begin{gathered}(\theta + A_Nx^N)y^{(N)} + A_{N-1}x^{N-1}y^{(N-1)} + A_{N-2}x^{N-2}y^{(N-2)} + \cdots + A_1xy' + A_0y = R(x) \\ y(0) = B_0 \\ y'(0) = B_1 \\ y''(0) = B_2 \\ \vdots \\ y^{N-1}(0) = B_{N-1}\end{gathered} \tag{3}$$

where $\theta \neq 0$.

The goal is to select an appropriate $R(x)$ and initial conditions such that the solution of this initial value problem is a polynomial. The recurrence relation of the homogeneous equation is first determined. Letting

$$y = \sum_{k=0}^{\infty} a_k x^k$$

leads to the recurrence relation

$$a_{k+N} = \frac{-a_k}{\theta}\left(\prod_{i=1}^{N}\frac{1}{k+i}\right)\left(A_0 + \sum_{j=1}^{N}\left(A_j\prod_{l=0}^{j-1}(k-l)\right)\right) \tag{4}$$

Let the auxiliary equation be defined as:

$$A_0 + \sum_{j=1}^{N}\left(A_j\prod_{l=0}^{j-1}(k-l)\right) = 0 \tag{5}$$

Since (5) does not depend upon θ, condition (2) also holds for (3). The existence of integers $k_i = n_i \geq 0,\ 0 \leq i < N$ that satisfy conditions (2) and (5) will ensure that the general homogeneous solution of (3) is a polynomial. However, for the vast majority of equations, the existence of such integers is most unlikely. We now deal with what can be done if we require that (3) only has a polynomial solution. It will turn out that it is possible to find functions $R(x)$ and initial conditions that guarantee that (3) only has polynomial solutions.

Derivation of Results:

Let's examine the negation of (2).

$$\exists r \ni 0 \leq r < N, \forall n \geq 0, n \bmod N \neq r \tag{6}$$

where n is a solution to (5).

The result of this statement is that there exists an infinite series homogenous solution that does not truncate into a polynomial. Fortunately, though, this solution depends entirely on the initial condition $y^{(r)}(0) = b_r$. Now suppose one chooses some integer M greater than or equal to zero such that $M \bmod N = r$. Set $R(x) = x^M$ in (3). The use of this $R(x)$ is justified by the following.

Suppose one wishes to make a particular solution to (3) equivalent to a finite portion (i.e., a polynomial of order M) of one of the infinite homogenous solutions.

Let

$$P(x) = a_M x^M + a_{M-N}x^{M-N} + a_{M-2N}x^{M-2N} + \cdots + a_r x^r$$

and substitute this expression into the original differential equation and solve for $R(x)$. Obviously, selecting this $R(x)$ will result in the desired particular solution. Now performing this substitution leads to

$$A_0 y = A_0(a_M x^M + a_{M-N}x^{M-N} + a_{M-2N}x^{M-2N} + \cdots + a_r x^r)$$

$$A_1 x y' = A_1(a_M(M)x^M + a_{M-N}(M-N)x^{M-N} + a_{M-2N}(M-2N)x^{M-2N} + \cdots + a_r r x^r)$$

$$A_2 x^2 y'' = A_2(a_M(M)(M-1)x^M + a_{M-N}(M-N)(M-N-1)x^{M-N} + \cdots + a_r r(r-1)x^r)$$

$$\vdots$$

$$A_r x^r y^{(r)} = A_r(a_M(M)(M-1)\cdots(M-r)x^M + \cdots + a_r r(r-1)\cdots(0)x^r)$$

$$\vdots$$

$$A_N x^N y^{(N)} = A_N(a_M(M)(M-1)\cdots(M-N+1)x^M + \cdots + a_r r(r-1)\cdots(0)\cdots(r-N+1)x^r)$$

$$\theta y^{(N)} = \theta(a_M(M)(M-1)\cdots(M-N+1)x^{M-N} + \cdots + a_r r(r-1)\cdots(0)\cdots(r-N+1)(0))$$

Clearly, the sum of the above lines must equal $R(x)$. Collecting like terms yields

$$a_M x^M(A_0 + A_1 M + A_2(M)(M-1) + \cdots + A_N(M)(M-1)\cdots(M-N+1))$$

$$x^{M-N}(a_M\theta(M)(M-1)\cdots(M-N+1) + a_{M-N}(A_0 + A_1(M-N) + \cdots + A_N(M-N)(M-N-1)\cdots(M-2N+1)))$$

$$x^{M-2N}(a_{M-N}\theta(M-N)\cdots(M-2N+1) + a_{M-2N}(A_0 + A_1(M-2N) + \cdots + A_N(M-2N)(M-2N-1)\cdots(M-3N+1)))$$

$$\vdots$$

$$x^r(a_{r+N}\theta(r+N)\cdots(r+1) + a_r(A_0 + A_1 r + A_2(r)(r-1) + \cdots + A_r r!))$$

The sum of these lines is also $R(x)$. Here comes the important observation. The recurrence relation (4) is equivalent to:

$$a_{k+N}\theta(k+n)\cdots(k+1)+a_k(A_0+A_1(k)+A_2(k)(k-1)+\cdots A_N(k)(k-1)\cdots(k-N+1)) = 0$$

This implies that all of the lines except for the one representing x^M are equal to zero. Thus,

$$R(x) = a_M(A_0+A_1M+A_2(M)(M-1)+\cdots A_N(M)(M-1)\cdots(M-N+1))x^M$$

Choosing a different coefficient before x^M will only effect the choice of the initial condition $y^{(r)}(0) = b_r$ that will eliminate the undesired homogeneous solution. The important result is that $R(x)$ only depends on x^M. For the sake of simplicity, choose the coefficient to be 1. Thus, $R(x) = x^M$.

The next step is to reexamine the homogeneous recurrence relation, but this time, in closed form. Let $k = pN+r$ for some integer $p > 0$.

$$a_k = a_{pN+r} = \frac{1}{\theta^p(k)!}\left(\prod_{i=0}^{p-1}\left(-A_0-\sum_{j=1}^{N}\left(A_j\prod_{l=0}^{j-1}(k-N(i+1)-l)\right)\right)\right)b_r \qquad (7)$$

Now set $k = M+N$ where $M = qN+r$. Because of the $R(x)$ term, the recurrence relation must be changed slightly when calculating a_{M+N}. After this is done, set $a_{M+N} = 0$ and solve for b_r:

$$a_{M+N} = a_{(q+1)N+r} = \frac{1}{\theta^{q+1}(M+N)!}\left(\prod_{i=0}^{q}\left(-A_0-\sum_{j=1}^{N}\left(A_j\prod_{l=0}^{j-1}(M+N-N(i+1)-l)\right)\right)\right)b_r+\frac{1}{\theta}\prod_{m=1}^{N}\frac{1}{M+m}$$

$$a_{M+N} = \frac{1}{\theta^{q+1}(M+N)!}\left(\prod_{i=0}^{q}\left(-A_0-\sum_{j=1}^{N}\left(A_j\prod_{l=0}^{j-1}(M-iN-l)\right)\right)\right)b_r+\frac{1}{\theta(M+N)(M+N-1)\cdots(M+1)} = 0$$

$$b_r = -\theta^q(qN+r)!\prod_{i=0}^{q}\left(-A_0-\sum_{j=1}^{N}\left(A_j\prod_{l=0}^{j-1}(M-iN-l)\right)\right)^{-1} \qquad (8)$$

Choosing this initial condition will ensure that the infinite series solution that depends on b_r will truncate into a polynomial of order M. This process can be repeated for all values of r for which there exists no solution $n \geq 0$ of (5) such that $n \bmod N = r$. If there are two or more infinite homogenous solutions, then $R(x)$ would be the sum of the various x^M's by the superposition principle.

Some Examples

Example 1*:*

Consider the following initial value problem:

$$(2+x^3)y''' + 5x^2y'' - 20xy' - 60y = R(x)$$
$$y(0) = b_0$$
$$y'(0) = b_1$$
$$y''(0) = b_2$$

Find an $R(x)$ and initial conditions that yield a finite polynomial solution.

Solution:

First, set up and evaluate the auxiliary equation to determine if any of the solutions are finite.

$$(r)(r-1)(r-2)+5(r)(r-1)-20(r)-60 = 0, \quad \text{Solution is}: \{r=5\},\{r=-4\},\{r=-3\}$$

The only useful root here is $r = 5$. Since $5 \bmod 3 = 2$, it is known that the homogeneous solution that depends on the initial condition $y''(0) = b_2$ is a polynomial. The other two solutions are not. As a result, a special $R(x)$ must be chosen that will yield only polynomial solutions of the initial value problem. Let's examine the solution that depends on $y(0) = b_0$. Pick a value of $M \geq 0$ such that $M \bmod 3 = 0$. One such value is $M = 9$. Let $R(x) = x^9$ for now. Now calculate the initial condition from (8).

The various variables are $r = 0, \theta = 2, A_0 = -60, A_1 = -20, A_2 = 5, A_3 = 1, N = 3, M = 9, q = 3$:

$$b_0 = -(2)^3((3)(3)+(0))!\prod_{i=0}^{3}(60+20(9-3i)-5(9-3i)(9-3i-1)-1(9-3i)(9-3i-1)(9-3i-2))^{-1} = -\frac{2}{195}$$

Now examine the solution that depends on $y'(0) = b_1$. Pick a value of $M \geq 0$ such that $M \bmod 3 = 1$. One such value is $M = 7$. Combining this result with that previously attained, let $R(x) = x^7 + x^9$. Now calculate the initial condition from (8). The various variables are $r = 1, \theta = 2, A_0 = -60, A_1 = -20, A_2 = 5, A_3 = 1, N = 3, M = 7, q = 2$:

$$b_1 = -(2)^2((2)(3)+(1))!\prod_{i=0}^{2}(60+20(7-3i)-5(7-3i)(7-3i-1)-1(7-3i)(7-3i-1)(7-3i-2))^{-1} = \frac{9}{440}$$

Now substitute $R(x)$ and the initial conditions into the original initial value problem.

$$(2+x^3)y''' + 5x^2y'' - 20xy' - 60y = x^7 + x^9$$
$$y(0) = \tfrac{-2}{195}$$
$$y'(0) = \tfrac{9}{440}$$
$$y''(0) = b_2$$

The solution becomes:

$$y(x) = -\frac{2}{195} + \frac{9}{440}x + \frac{1}{2}b_2x^2 - \frac{2}{39}x^3 + \frac{3}{88}x^4 + \frac{3}{8}b_2x^5 - \frac{7}{390}x^6 + \frac{1}{220}x^7 + \frac{1}{624}x^9$$

which is a finite polynomial as desired. END

Classical Equations

This technique is also particularly useful in physics and engineering. The well-known equations of Hermite, Legendre, and Chebyshev are all special cases of (3). In each of these cases, solving the auxiliary equation yields exactly one positive integer root that matches the parameter that is passed to the respective equation. This indicates that these equations each have exactly one polynomial solution in the homogenous case. These polynomials form the set of Hermite, Legendre, and Chebyshev polynomials. However, since these equations are of order 2, they also yield a second homogenous solution that is not a polynomial. In no case can they be polynomials since the auxiliary equations do not yield second roots which are integers greater than or equal to zero, and thus, condition (6) is satisfied. The only exception is the Chebyshev equation with a parameter of zero. Zero becomes a repeated root, but then, for $r = 1$, there is no root n such that $n \bmod 2 = r$, and thus, condition (6) is still satisfied.

One is left with the task of generating a polynomial solution which is done as previously shown. Choose $R(x) = x^M$. If the parameter of the equation is even, choose M to be odd. Similarly, if the parameter of the equation is odd, choose M to be even. This ensures that $M \bmod 2 \neq k \bmod 2$ where k represents the parameter. Then, selecting an appropriate value of $y^{(M \bmod 2)}(0) = b_{M \bmod 2}$ will ensure a polynomial solution. Now, let's consider an illustrative example:

Example 2:

Suppose a physicist conducts an experiment where a particle's position can be determined by Legendre's equation with a parameter of 3:

$$(1-t^2)\tfrac{d^2x}{dt^2} - 2t\tfrac{dx}{dt} + 12x = R(t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

The physicist has an apparatus that will act on the particle with a force equivalent to the time raised to some positive integer power. Supposing that the physicist wants to restrict the particle's motion to a fifth degree polynomial, what setting should s/he choose for the forcing apparatus? In addition, what initial position and velocity should s/he choose?

Solution:

Substituting into the auxiliary equation,

$$-r(r-1) - 2r + 12 = 0, \quad \text{Solution is : } \{r = -4\}, \{r = 3\}$$

Clearly, the only useful root is $r = 3$, which matches the parameter passed to the Legendre equation. It follows that the third order Legendre polynomial $\frac{5}{2}x^3 - \frac{3}{2}x$ is one of the homogeneous solutions. As shown previously, the second solution is not a polynomial. In order to meet the physicist's specifications, the second solution must be reduced to a fifth degree polynomial. Suppose $M = 5$ is chosen. Unfortunately, $5 \bmod 2 = 3 \bmod 2 = 1$, which indicates there is a conflict. M cannot be 6 since the second solution would become a polynomial of order 6, which would violate the specifications. Thus, M should be set to 4. Let $R(t) = t^4$. Next, the initial condition $x(0) = x_0$ is chosen according to (8). Here, $r = 0, \theta = 1, A_0 = 12, A_1 = -2, A_2 = -1, N = 2, M = 4, q = 2$:

$$x_0 = -(1)^2((2)(2) + (0))! \prod_{i=0}^{2}(-12 + 2(4 - 2i) + 1(4 - 2i)(4 - 2i - 1))^{-1} = -\frac{1}{24}$$

Substituting the results back into the initial value problem yields:

$$(1 - t^2)\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 12x = t^4$$
$$x(0) = \frac{-1}{24}$$
$$x'(0) = v_0$$

which has the solution:

$$x(t) = -\frac{1}{24} + \frac{1}{4}t^2 - \frac{1}{8}t^4 - \frac{1}{3}v_0(5t^3 - 3t)$$

To answer the original question, the physicist would choose the setting 4 and select a starting position equal to $\frac{-1}{24}$. The initial velocity would be left as arbitrary. END

Concluding Remarks

The results shown can also be applied to any linear differential equation that has an ordinary point at $x = 0$. Consider the differential equation

$$L[y] = R(x) \tag{9}$$

which is of order N and has N initial conditions. Suppose one solution is $g_i(x)$. Define $p_i(x)$ as a polynomial that contains a portion of the infinite series expansion of $g_i(x)$ about $x = 0$ if $g_i(x)$ isn't a polynomial, or 0 if $g_i(x)$ is a polynomial. If we let $p_i(x)$ be a particular solution, then we can generate a special $R(x)$ by using the following formula:

$$R(x) = \sum_{i=1}^{N} L[p_i(x)] = L[P(x)]$$

where $P(x)$ represents the entire particular solution.

Now that the desired particular solution has been obtained, the infinite homogenous solutions must be eliminated by choosing the initial conditions. From the general theory of linear differential equations, it is known that the general solution of (9) is:

$$y = P(x) + \sum_{i=0}^{N} c_i y_i(x)$$

where $y_i(x)$ are n linearly independent homogenous solutions.

Now consider the vector $\vec{y}$ which contains y and all its derivatives up to $y^{(N-1)}$. $\vec{y}(0)$ represents the desired initial conditions. Let $\vec{y_i}$ and $\vec{P}$ be defined in a similar fashion except for the fact that they apply to the individual homogeneous solutions and the particular solution respectively. If a system of equations is set up to determine the initial conditions, the result becomes:

$$\vec{y}(0) = \vec{P}(0) + \sum_{i \in K} c_i \vec{y_i}(0)$$

where the set K contains all values k such that $y_k(x)$ is a polynomial homogeneous solution. Here, the various c_i's are chosen arbitrarily.

While this method would actually be easier to use on (3), it comes with a major disadvantage. The method cannot be used unless the homogeneous solutions are known. When using the main technique outlined in this paper, the only requirement is knowing whether a given homogeneous solution is a polynomial or not, which can easily be determined by the roots of the auxiliary equation (5).

The use of the technique presented is important to science and engineering since many physical phenomena can be mathematically described by special cases of (1). In addition, it is particularly useful in the teaching of mathematics. When teaching the method of series solutions, it is often helpful for a student to see an example in which one of the series solutions truncates to a polynomial solution. If forcing terms are included, then students must analyze how the method of series solutions applies to nonhomogeneous differential equations. Furthermore, the technique can be used when it is desired to slow down the growth of a given solution. For example, a rapidly increasing exponential solution could be reduced to polynomial growth of some given order. In fact, the uses are endless, with more to come as mathematics and science continue to develop.

[1] "Nth-order differential equations with finite polynomial solutions" by Gabriel B. Costa and Lawrence E. Levine, Int. J. Math. Educ. Sci. Tech. 29, No 6, 1998 pp.911-914.