

Polynomial Solutions of the Laguerre Equation and Other Differential Equations Near a Singular Point

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Abstract

It can be shown that if a differential equation is analytic near a point, then it is always possible to select a forcing term along with initial conditions that will ensure the solution to the new nonhomogeneous equation is a polynomial that is the finite, truncated portion of the (infinite) series solution of the original equation. It turns out that this result can be extended to expansions about a singular point. The conditions under which such a polynomial truncation can be accomplished about a singular point are presented in the appendix. A brief algorithm is described that enables one to choose the appropriate forcing term and initial conditions. Following this, an example involving Laguerre's Equation is presented.

Introduction

It is often of interest to mathematicians to take the solution of a differential equation about a point and truncate its series expansion into a polynomial by means of transforming the original equation into something that is similar, but nonhomogeneous. Consider the basic n th order linear differential equation:

$$L[y] = 0 \tag{1}$$

For the sake of simplicity, assume that a polynomial solution to $L[y] = R(x)$ is desired near the point $x = 0$. (The analysis is extremely similar at any other point.) Suppose that $y_i(x)$ is a solution of (1). Let $y_i^\diamond(x)$ be a polynomial truncation of $y_i(x)$, if one exists. We make this truncation the particular solution by selecting the forcing term:

$$R(x) = L[y_i^\diamond(x)] \quad (2)$$

Then we select initial conditions that will set the arbitrary constant in front of $y_i(x)$ to zero. This can be repeated for every non-polynomial solution of (1). By superposition, the final forcing term is simply the sum of the respective forcing terms derived for each non-polynomial solution. This process will be analyzed more carefully in the following section.

Forcing Terms and Initial Conditions

Once again, consider the linear n th order differential equation $L[y] = 0$ at $x = 0$. Let $\mathcal{Y} = \{y_i(x)\}$ be the set of n linearly independent solutions of this equation. Let $\mathcal{P} \subseteq \mathcal{Y}$ be the set of polynomial solutions of the equation. Let $\mathcal{P}^* = \mathcal{Y} \setminus \mathcal{P}$, i.e. the set of all non-polynomial solutions of the equation. With these conventions in mind, the following is a general method that can be used to create a polynomial truncation solution of this equation. First, $\forall y(x) \in \mathcal{P}^*$, select a $y_i^\diamond(x)$, provided that one exists. If one doesn't exist, then it is useless trying to find a polynomial solution. The best one can do is simply eliminate that solution and have no polynomial representation of it. If despite this setback, a polynomial solution is still desired, then choose $y_i^\diamond(x) = 0$. Next, construct the particular solution $p(x)$. Choose it as follows:

$$p(x) = \sum_{i \in \mathcal{P}^*} y_i^\diamond(x) \quad (3)$$

Then the forcing term

$$R(x) = L[p(x)] \quad (4)$$

can be selected. The next step is to eliminate the non-polynomial homogeneous solutions. This is done by selecting the i th initial condition as

$$y^{(i)}(0) = \sum_{k \in \mathcal{P}} (c_k y_k^{(i)}(0)) + p^{(i)}(0) \quad (5)$$

where the c_k 's are arbitrary constants and $0 \leq i < n$. This selection ensures that the arbitrary constants before the non-polynomial homogeneous solution are all zero. Thus the complete solution becomes a polynomial, and we no longer need worry about any of the homogeneous solutions not being well-defined at $x = 0$.

Laguerre's Equation

We now demonstrate the technique illustrated in the paper by working through an example involving Laguerre's Equation, which is

$$xy'' + (1 - x)y' + \lambda y = 0 \quad (6)$$

where λ is a nonnegative integer. We will find a forcing term and initial conditions to ensure that its solution is a polynomial. It is well known that one solution of Laguerre's Equation is

$$y_1(x) = L_\lambda(x), \quad (7)$$

which is known as the Laguerre polynomial of order λ . Since $x = 0$ is a regular singular point, we may use the Method of Frobenius to show that the second linearly independent solution takes the form:

$$y_2(x) = L_\lambda(x) \ln(x) + \sum_{n=1}^{\infty} b_n x^n \quad (8)$$

where the b_n 's are coefficients that must be determined.¹

Fortunately, for the remainder of the analysis, the exact values of the b_n 's will not be important. It is evident from (7) and (8) that $\mathcal{P} = \{y_1(x)\}$ and $\mathcal{P}^* = \{y_2(x)\}$. The next step is to find a polynomial truncation of $y_2(x)$. The term with the logarithm cannot have a polynomial truncation. (See the end of the Appendix for more about this.) The reason for this is outlined in the appendix. Thus, the following particular solution should be selected:

$$p(x) = y_2^\diamond(x) = \sum_{k=1}^n b_k x^k \quad (9)$$

¹For a more detailed discussion of the solution to Laguerre's Equation, consult any general differential equations text. One example is Rainville, Earl D. *Elementary Differential Equations*, 8th Edition. Prentice Hall, 1996. ISBN: 0135080118.

The next step is to determine the forcing term $R(x)$. By examining (6) and (9), one sees that $R(x) = L[p(x)]$ is analytic. Therefore we rewrite the nonhomogeneous equation as follows:

$$xy'' + (1-x)y' + \lambda y = \sum_{k=0}^{\infty} r_k x^k \quad (10)$$

For the sake of simplicity, rewrite $p(x)$ as:

$$p(x) = \sum_{k=0}^{\infty} a_k x^k \quad (11)$$

where $a_0 = 0$, $a_i = b_i$ if $0 < i \leq n$, and $a_i = 0$ if $i > n$. Now substitute this expression for y in (10). After some algebra the following equation is obtained:

$$a_1 + a_0\lambda - r_0 + \sum_{k=1}^{\infty} [(a_{k+1}(k+1)^2 + a_k(\lambda - k) - r_k) x^k] = 0 \quad (12)$$

The values of the a_k 's are known. They will be used to solve for the r_k 's. Starting with the constant term outside of the summation, it is clear that

$$r_0 = b_1 \quad (13)$$

since $a_0 = 0$ and $a_1 = b_1$. When $0 < i < n$, $a_i = b_i$ and $a_{i+1} = b_{i+1}$. Thus,

$$r_i = b_{i+1}(i+1)^2 + b_i(\lambda - i) \quad \mathbf{if} \quad 0 < i < n \quad (14)$$

When $i = n$, $a_n = b_n$ but $a_{n+1} = 0$. This implies that

$$r_n = b_n(\lambda - n) \quad (15)$$

Finally, if $i > n$, then $a_n = a_{n+1} = 0$. Thus,

$$r_i = 0 \quad \mathbf{if} \quad i > n \quad (16)$$

(13)-(16) give the values of the coefficients of the polynomial that make up the forcing term $R(x)$. All that is left to do is calculate the initial conditions. Applying equation (5) to this problem yields:

$$\begin{aligned} y(0) &= c_1 L_\lambda(0) \\ y'(0) &= c_1 L'_\lambda(0) + b_1 \end{aligned} \quad (17)$$

It may also be of interest to choose these initial conditions in a slightly different form. Let A be an arbitrary constant and set it equal to $c_1 L_\lambda(0)$. Also, use the substitution $b_1 = 1 + 2\lambda$ as shown in (20). This yields:

$$\begin{aligned} y(0) &= A \\ y'(0) &= A \frac{L'_\lambda(0)}{L_\lambda(0)} + 2\lambda + 1 \end{aligned} \tag{18}$$

This choice of initial conditions emphasizes the fact the $y(0)$ can be chosen completely arbitrarily for a polynomial truncation solution to result.

Conclusion

The results that have been presented establish the conditions that are required in order to construct a nonhomogeneous differential equation with a solution that is a polynomial truncation of any homogeneous linear differential equation, regardless of whether the desired solution is considered about an ordinary or singular point. Then, a method of actually constructing these equations and solutions was presented and demonstrated. This work is a continuation of research concerning the existence of polynomial solutions of differential equations. In [1] a thorough analysis of polynomial solutions of analytic equations was presented. In [2], using a technique similar to those employed in [1] and this paper, complete polynomial solutions of the well-known equations of Hermite, Legendre, and Chebyshev in their nonhomogeneous forms were derived. This paper concludes work done on ordinary linear differential equations. Future research will transcend the realm of linearity. Applications to partial differential equations will also be considered. Hopefully, more generalizations will be discovered, which will increase our awareness of the nature of polynomials with respect to differential equations.

Appendix

A fundamental question in this analysis is whether $f^\diamond(x)$, a polynomial truncation of some function $f(x)$, exists. As a side note, $f^\diamond(x)$ is rarely unique for a given $f(x)$. If $f(x)$ is analytic at $x = 0$, then the problem is

trivial. Since $f(x)$ is analytic, it can be rewritten as:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i \quad (19)$$

We can simply choose an $f^\diamond(x)$ as follows:

$$f^\diamond(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n \quad (20)$$

where $n \in \mathbb{N}$. When $f(x)$ fails to be analytic, the problem is trickier. To analyze this situation we must split up $f(x)$ into analytic and non-analytic components, i.e.

$$f(x) = a(x) + b(x) \quad (21)$$

where $a(x)$ is analytic and $b(x)$ is not analytic. If $a(x) \neq 0$, then it is possible to select $f^\diamond(x) = a^\diamond(x)$ where $a^\diamond(x)$ is selected as shown in (20). The question that remains is how one goes about finding the analytic component of a function. One way is to consider the series expansion of $f(x)$ as a continuous, as opposed to discrete, entity. We must determine the spectrum of powers of x contained in $f(x)$ by using an integral transform, which is actually the Fourier Transform in disguise.

Theorem: Let $f(x)$ be a function. The spectrum of powers of x is given by the expression

$$p(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(e^{ix}) e^{-inx} dx = \frac{1}{2\pi} \mathcal{F} [f(e^{ix})] \quad (22)$$

provided that the integral exists.

Proof. Consider a function that has a Laurent expansion. That is, it takes the form:

$$f(z) = \sum_{n=-\infty}^{\infty} P(n) z^n \quad (23)$$

We want a summation that includes all real values of n to take care of the case where $f(x)$ does not have a Laurent expansion. After applying a limiting process and introducing a differential, the summation becomes the following integral:

$$f(z) = \int_{-\infty}^{\infty} p(n) z^n dn \quad (24)$$

Using the substitution $z = e^{ix}$, we see that (24) is really an inverse Fourier Transform. Hence,

$$f(e^{ix}) = \int_{-\infty}^{\infty} p(n)e^{inx} dn = 2\pi \mathcal{F}^{-1}[p(n)] \quad (25)$$

Taking Fourier Transforms of both sides and dividing by 2π yields

$$p(n) = \frac{1}{2\pi} \mathcal{F}[f(e^{ix})] \quad (26)$$

establishing the result. \square

Using this information, we can now show how to find the functions $a(x)$ and $b(x)$. Suppose the spectrum function $p(n)$ takes the form:

$$p(n) = \sum_{k \in \mathcal{K}} (a_k \delta(n - k)) + r(n) \quad (27)$$

where $\delta(n)$ is the Dirac Delta Function, \mathcal{K} is the set of all real numbers which correspond to nonzero values of a_k and $r(n)$ represents the remainder after the delta functions have been eliminated. Then we define $a(x)$ and $b(x)$ to be:

$$a(x) = \sum_{k \in \mathcal{K} \cap \mathbb{N}} a_k x^k \quad (28)$$

and

$$b(x) = \sum_{k \in \mathcal{K} \setminus \mathbb{N}} a_k x^k + \int_{-\infty}^{\infty} r(n)x^n dn \quad (29)$$

where $\mathcal{K} \setminus \mathbb{N}$ refers to the set of all elements of \mathcal{K} that are not natural numbers. In essence, for a polynomial truncation to exist, the spectrum of powers of x must have at least one “spike” at a nonnegative integer. If there is no spike at any nonnegative integer power n of x , then there is no justification for including an x^n term in the polynomial solution. Although it is possible to make any polynomial the solution of a differential equation, we need this polynomial to be a polynomial truncation of the function so that the polynomial solution of the differential equation has some of the properties of the homogeneous non-polynomial solution.

In the main text, it was mentioned without explanation that the function $\ln(x)$ does not have a polynomial truncation. To show this fact, consider

the power series spectrum of $\ln(x)$ about zero.

$$p(n) = \frac{1}{2\pi} \mathcal{F} [\ln(\exp(ix))] = \frac{i}{2\pi} \mathcal{F} [x] = -\frac{d}{dn} \delta(n) \quad (30)$$

There is a derivative of the delta function at zero but no pure delta functions at any nonnegative integers. Thus the $\ln(x)$ term has no analytic component that can contribute to $y_2^\diamond(x)$. Also, multiplying $\ln(x)$ by any of the terms of $L_\lambda(x)$ will only horizontally shift the spectrum function $p(n)$. As a result, we can only have the polynomial truncation shown in (9).

[1]: *Polynomial Solutions of Nth Order Differential Equations* by Lawrence E. Levine and Ray Maleh. *Int. J. Math. Educ. Sci. Tech.* 33, No 6, 2002 pp. 898-906.

[2]: *Polynomial Solutions of the Classical Equations of Hermite, Legendre, and Chebyshev* by Lawrence E. Levine and Ray Maleh. To appear in *Int. J. Math. Educ. Sci. Tech.*