## Ma 116 Calculus II Quiz I 3 October 2006

**1** [20 **pts.**] Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converge? Does the series converge absolutely? Justify your answers.

Solution: The series converges. It is an alternating series, $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, and as n increases, the denominator increases and hence $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$.

The series does not converge absolutely by comparison with the harmonic series, i.e. $\frac{1}{n} < \frac{1}{\ln n}$ for $n$ larger than 2.

**2 [20 pts.]** Does the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge? Does the series converge absolutely? Justify your answers. .

.Solution: We use the integral test, noting that the integral will be in the form $\int \frac{du}{u}$ with $u = \ln x$.

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x} dx &= \lim_{L\to\infty} \int_2^{L} \frac{1}{x \ln x} dx = \lim_{L\to\infty} \int_{\ln 2}^{\ln L} \frac{du}{u} \\ &= \lim_{L\to\infty} [\ln \ln L - \ln \ln 2] = \infty \end{aligned}$$

So the integral diverges and hence the series diverges. Since the series does not converge, it cannot converge absolutely.

**3 [20 pts]** Let $f(x) = \frac{1}{2}(e^x + e^{-x})$. Find the Maclaurin series (i.e. the Taylor series about $x = 0$) for $f(x)$. Show the first three non-zero terms and the general term for the series.

Solution:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ f(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) & f(0) = \frac{1}{2}\left(e^0 + e^{-0}\right) = \frac{1}{2}(1+1) = 1 \\ f'(x) &= \frac{1}{2}\left(e^x - e^{-x}\right) & f'(0) = \frac{1}{2}\left(e^0 - e^{-0}\right) = \frac{1}{2}(1-1) = 0 \\ f''(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) & f''(0) = \frac{1}{2}\left(e^0 + e^{-0}\right) = \frac{1}{2}(1+1) = 1 \end{aligned}$$

So derivatives now repeat and alternate between 1 and 0 with the even order derivatives all equal to 1 and the odd ones equal to zero. Hence.

$$f(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{1}{(2n)!}x^{2n} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

**4a [30 pts.]** Find all values of $x$ for which the following series converges. What is the radius of convergence of the series?

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{\sqrt{n+1}} (x-1)^n .$$

Solution: For this case $c_n = \frac{(-2)^n}{\sqrt{n+1}}$. The convergence radius is

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \frac{1}{2}.$$

So, the series converges for $x \in \left(\frac{1}{2}, \frac{3}{2}\right)$.

To obtain this result, we could also apply the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{\sqrt{n+2}} \frac{\sqrt{n+1}}{(-2)^n} \frac{(x-1)^{n+1}}{(x-1)^n} \right| &= \lim_{n\to\infty} \left| \frac{2\sqrt{n+1}}{\sqrt{n+2}} (x-1) \right| \\ &= 2\,|x-1| \end{aligned}$$

So the series converges when $2\,|x-1| < 1$, i.e for $-\frac{1}{2} < x-1 < \frac{1}{2}$, or $\frac{1}{2} < x < \frac{3}{2}$, or $x \in \left(\frac{1}{2}, \frac{3}{2}\right)$. It remains to check the endpoints of the interval of convergence. When $x = \frac{1}{2}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{\sqrt{n+1}} \left(\frac{-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}.$$

This is the $p$-series with $p = 1/2$ which is divergent. At the other endpoint $x = \frac{3}{2}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{\sqrt{n+1}} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which is a convergent alternating series. Hence, the interval of convergence is $\frac{1}{2} < x \leq \frac{3}{2}$, or $x \in \left(\frac{1}{2}, \frac{3}{2}\right]$.

**4b** [10 *pts.*] Find all values of x for which the following series converges. What is the radius of convergence of the series?

$$\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n+1}} (x-1)^n .$$

Solution: Removing the negative sign has no effect inside the ratio test, so that part of the analysis from problem 4a is unchanged. At the endpoints, the convergent and divergent series swap places, so the radius of convergence is still $\frac{1}{2}$ and the interval of convergence is $\frac{1}{2} \leq x < \frac{3}{2}$, or $x \in \left[\frac{1}{2}, \frac{3}{2}\right)$.