1. [20 pts] Given function $f(x, y) = (x + 2y)^2 e^{1+y}$, find

- gradient $f(x, y)$ at point $A(3, -1)$.
- the directional derivative of $f(x, y)$ at $A(3, -1)$ in the direction of point $B(6, -5)$

Solution. $\nabla f(x, y) = \langle 2(x + 2y)e^{1+y}, 4(x + 2y)e^{1+y} + (x + 2y)^2 e^{1+y} \rangle$.
Then $\nabla f(3,-1) = \langle 2,5 \rangle$.

$\vec{AB} = \langle 3, -4 \rangle \Rightarrow \vec{u} = \frac{\vec{AB}}{|\vec{AB}|} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$.
Then $D_{\vec{u}} f(3, -1) = \nabla f \circ \vec{u}$

$$= \frac{3}{5} \cdot 2 - \frac{4}{5} \cdot 5 = -\frac{14}{5}.$$

2. [30 pts] (a) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at point $P(0, 1, 1)$ if $y^2 e^{xy} + yz + ze^{xz} = 3$.
Solution. If $F(x, y, z) = 0$ then

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}.$$

In our case $F(x, y, z) = y^2 e^{xy} + yz + ze^{xz} - 3$. Then

$$F'_x = y^3 e^{xy} + z^2 e^{xz}, \quad F'_y = (2y + xy^2)e^{xy} + z, \quad F'_z = y + e^{xz} + xze^{xz}.$$

At point $P(0, 1, 1)$ we obtain

$$F'_x(P) = 2, \quad F'_y(P) = 3, \quad F'_z(P) = 2.$$

Consequently, at point P

$$\frac{\partial z}{\partial x} = -1, \qquad \frac{\partial z}{\partial y} = -\frac{3}{2}.$$

(b) Find the equation of tangent plane to the surface $y^2 e^{xy} + yz + ze^{xz} = 3$ at point $P(0, 1, 1)$.

Solution. Using the formula $z - z_0 = z'_x(x - x_0) + z'_y(y - y_0)$, we obtain
$z - 1 = -(x - 0) - \frac{3}{2}(y - 1)$, or $2x + 3y + 2z - 5 = 0$.
Answer: $2x + 3y + 2z = 5$.
Remark: the same result may be obtained by using another formula for tangent plane: $F'_x(x - x_0) + F'_y(y - y_0) + F'_z(z - z_0) = 0$. Then instantly
$2(x - 0) + 3(y - 1) + 2(z - 1) = 0$.

3. [20 pts] Find the critical points for the function $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$. Which of them are local maxima, minima and saddle points? Justify your answer.
Solution. $f(0, 0) = 0$, local minimum; $f(1, -1)$, saddle point.

4. [30 pts] Evaluate $\iint_D x\sqrt{y^2 - x^2}dA \qquad$ if $D = \{(x, y) : \ 0 \le x \le 1, \ x \le y \le 1\}$.

Solution.

$$\iint_D x\sqrt{y^2 - x^2}dA = \int_0^1 \int_0^y x\sqrt{y^2 - x^2}dx\, dy = \int_0^1 \left[-\frac{1}{3}(y^2 - x^2)^{3/2}\right]\Big|_{x=0}^{y} dy = \frac{1}{12}.$$