Universal Groups of Prees

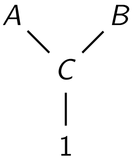
Robert Gilman

Stevens Institute of Technology

Cornell University
May 10, 2015

What is a pree?

The canonical example is an amalgam of groups with partial multiplication $aa' = a'', bb' = b''$.



The partial multiplication has an identity, inverses, and is associative where possible.

This pree embeds in its universal group

$$\langle A \cup B \mid aa' = a'', bb' = b'' \rangle.$$

Its Cayley diagram is essentially the link of the presentation complex.

Definition

A pree is a set P with a partial binary operation such that

1. There is an identity, i.e., an element $1 \in P$ such that for all $a \in P$, the products $1a$ and $a1$ are defined and are both equal to a.
2. Each $a \in P$ has a 2-sided inverse.
3. If ab and bc are defined, then $(ab)c$ is defined if and only if $a(bc)$ is defined, in which case $(ab)c = a(bc)$.

It is not hard to show that inverses are unique in a pree.

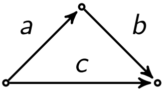
Further if $ab = c$, then $b = a^{-1}c$ etc.

Every group is a pree.

Every finitely presented group is the universal group of a finite pree.

Triangles

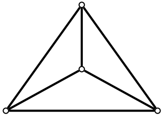
Each product $ab = c$ defined may be represented by a triangle.



Each valid triangle yields six defined products

$$ab = c \quad a^{-1}c = b \quad bc^{-1} = a^{-1} \quad \cdots$$

The geometric meaning of the associative law: if three valid triangles fit around a common vertex, then their perimeter is valid too.



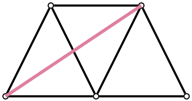
Embeddability

The universal group, $U(P)$, of a pree, P, has generators P and relations $ab = c$ for all products defined by the partial multiplication.

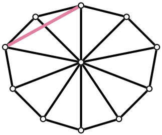
The question of whether a finite pree embeds in its universal group is undecidable. [T. Evans 1951]

Baer introduced several axioms sufficient for embeddability,

Stallings[1971] used Axiom P5 to define pregroups.



Axiom P5



Axiom S

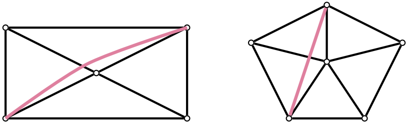
Several other axioms have been considered. See [Gaglione, Lipschutz, Spellman, 2012].

Theorem

Universal groups of finite prees satisfying Axiom P5 or Axiom S are virtually free.

Apply the characterization of virtually free groups as groups with context free word problem [Muller, Schupp 1983]

We consider prees satisfying two axioms introduced by Baer and illustrated below.



Axioms A4 and A5

Theorem (RG 2014)

If P is a pree satisfying Axioms A(4) and A(5), then P embeds in $U(P)$, and the multiplication in P is induced by the multiplication in $U(P)$.

The Cayley diagram of a pree has the elements of $P - \{1\}$ as its vertices and edges corresponding to valid products $xy = z$ with $x, y, z \in P - \{1\}$. Each such product determines an edge from x to z with label y.

The Cayley diagram is essentially the link of the vertex of the presentation complex corresponding to the pree.

Axioms A4 and A5 are a weaker form to the small cancellation condition C3-T6.presentation complex

There are many such variations by e.g,. A. Juhasz, J. McCammond and D. Wise, Yu. Ol'shanskĭ, U. Weiss.

Theorem (RG, 2014)

If P is a finite pree satisfying Axioms A(4) and A(5), then $U(P)$ is biautomatic.

The theorem generalizes [Gersten and Short 1990] for C(3)-T(6) small cancellation presentations with all pieces of length 1.

Finite C(3)-T(6) groups are cyclic, but all finite groups are universal groups of A4-A5 prees (namely themselves).

Theorem (RG)

Let P satisfy A4 and A5. The universal group of P has a regular geodesically perfect rewriting system.

[Diekert, Duncan, Miasnikov, 2010] extend the Knuth-Bendix procedure to a procedure for geodesically perfect rewriting system as follows

A geodesically perfect rewriting system contains

1. Length-reducing reductions which rewrite any word to an equivalent geodesic word;
2. Length-preserving reductions which rewrite any two equivalent geodesic words to each other.

Example 1.

$G = \langle a, b \mid ab = ba \rangle$

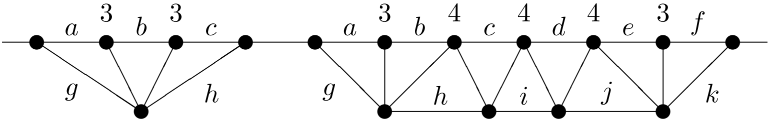
Define a pree P by triangulating the relations.

$$ab = c = ba$$

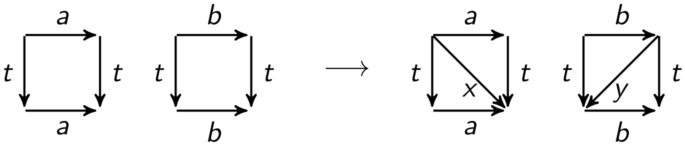
G is the universal group of the pree $P = \{1, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$ and all products corresponding to the relations above.

Check associativity and the two additional axioms by looking for cycles of length at most 6 in the Cayley diagram of P

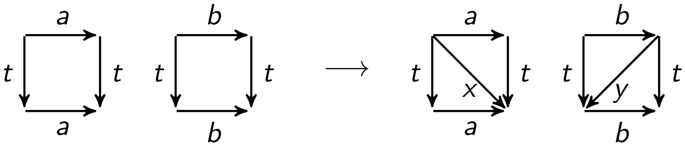
The geodesically perfect rewriting system is regular and infinite



Example 2. $G = \langle a, b, t \mid [a,t] = [b,t] = 1 \rangle$

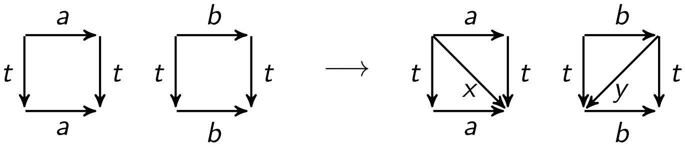


Example 2. $G = \langle a, b, t \mid [a, t] = [b, t] = 1 \rangle$



The triangles determine a pree.

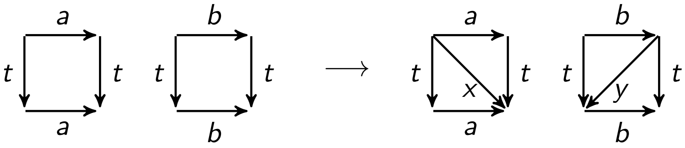
Example 2. $G = \langle a, b, t \mid [a,t] = [b,t] = 1 \rangle$



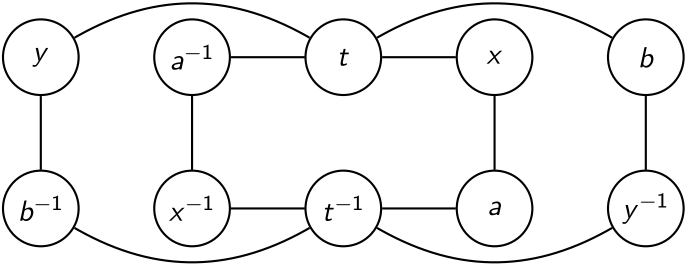
The triangles determine a pree.

	a	a^{-1}	b	b^{-1}	t	t^{-1}	x	x^{-1}	y	y^{-1}
a		1			x			t^{-1}		
a^{-1}	1					x^{-1}	t			
b				1		y^{-1}			t	
b^{-1}			1		y					t^{-1}
t	x			y		1		a^{-1}		b
t^{-1}		x^{-1}	y^{-1}		1		a		b^{-1}	
x		t				a		1		
x^{-1}	t^{-1}				a^{-1}		1			
y			t			b^{-1}				1
y^{-1}				t^{-1}	b				1	

Example 2. $G = \langle a, b, t \mid [a, t] = [b, t] = 1 \rangle$



G is biautomatic if there are no cycles of length < 6 in the Cayley diagram of the partial multiplication table.

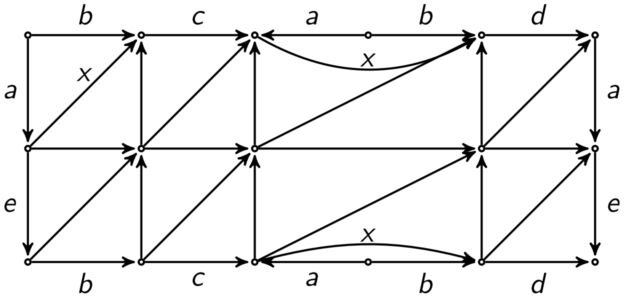


One-relator groups

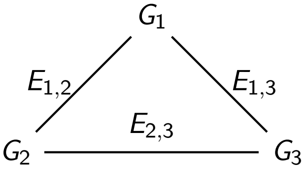
Conjecture. Every one-relator group whose relator is a commutator is automatic.

Example 3. $G = \langle a, b, c, d, e \mid [ae, bcabd] = 1 \rangle$

It suffices that $G * F_3 = \langle a, b, c, d, e, p, q, r \mid [ae, bcabd] = 1 \rangle$ is automatic.



Triangles of groups



Triangles of groups: Gersten and Stallings, Corson, Bridson and Haefliger.

Non-positively curved triangles with finite vertex groups have biautomatic universal groups [Floyd and Parry 1995].

Tits Alternative: [Howie, Kopteva 2006], [Cuno, Lehnert 2014]

Positively curved triangles: [Chermak 1995], [Allcock 2012]

A triangle of groups is a pree.

Triangles of prees

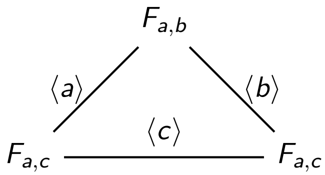
Theorem (RG)

Let T be a triangle of prees satisfying A4 - A5 and such that edge prees have the induced multiplication from each of their vertex prees and satisfy the following conditions.

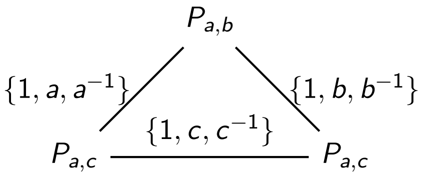
1. *The distance between any two nontrivial elements of an edge pree is ≥ 3 in each of its vertex prees.*
2. *The distance between any two nontrivial elements in distinct edge prees of a vertex pree is not 2.*

Then T also satisfies A4-A5.

Example 4



$F_{x,y}$ is the universal group of the pree $P_{x,y} = \{1, x, x^{-1}, y, y^{-1}\}$ with no products defined.



Example 5

Vertex groups $\langle a, b, c \mid a^b = [a, c] \rangle$

Edge prees: $\{1, a, a^{-1}\}$ and $\{1, b, b^{-1}\}$.

The triangle of universal groups of edge and vertex prees has all angles 0. The universal group of the triangle is biautomatic but not hyperbolic.