FE610 Stochastic Calculus for Financial Engineers

Lecture 2. A Primier on the Arbitrage Theorem

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Outline

Arbitrage: in its simplest form, it means taking simultaneous positions in different assets so that one guarantees a riskless profit higher than the riskless return given by U.S. Treasury bills. Two types of arbitrage opportunities:

- One can make a series of investments with no current net commitment, yet expect to make a positive profit. (portfolio: short-sell a stock; buy call options on the same security)
- A portfolio can ensure a negative net commitment today, while yielding non-negative profits in the future.

Determining arbitrage-free (*correctly priced*) prices is at the center of valuing derivative assets.

- A derivative house decides to engineer a *new financial product*.
- Risk managers would like to measure the risks associated with their portfolios.
- A treasurer need to know the current market value of a nonliquid asset for which no trades have been observed lately.
- Significant differences between observed and arbitrage-free values might indicate excess profit opportunities.

- **Asset Prices:**

  The index t will represent time. Securities such as options, futures, forwards, and stocks will be represented by a *vector* of asset prices denoted by S_t.

$$S_t = \begin{bmatrix} S_1(t) \\ \vdots \\ S_N(t) \end{bmatrix} \tag{1}$$

  Here, $S_1(t)$ may be riskless borrowing or lending, $S_2(t)$ may denote a particular stock, $S_3(t)$ may be a option written on this stock, $S_4(t)$ may represent the corresponding put option, and so on.

  In discrete time, securities prices can be expressed as $S_0, S_1, ..., S_t, S_{t+1}, ...$. However, in continuous time, the t subscript can assume any value between zero and infinity. We formally write this as

- **Asset Prices (continued):**

$$t \in [0, \infty). \tag{2}$$

In general, 0 denotes the *initial point*, and t represents the *present*. If we write

$$t < s, \tag{3}$$

then s is meant to be a future date.

- **States of the World:**

We let vector W denote all possible *state of the world*,

$$W_t = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_K \end{bmatrix} \tag{4}$$

where each ω_i represents a distinct outcome that my occur. These states are *mutually exclusive*.

In general, financial assets will have different values and give different payouts at different states of the world ω. It is assumed that there are a finite number K of such possible states.

- **Returns and Payoffs:**

  We let d_{ij} denote the number of units of account paid by one unit of security i in state j. These payoffs will have two components:

  - Capital gains or losses.
  - Dividends or coupon interest payments

$$D = \begin{bmatrix} d_{11} & \dots & d_{1K} \\ \vdots & \vdots & \vdots \\ d_{N1} & \dots & d_{NK} \end{bmatrix} \tag{5}$$

  If current prices of all assets are nonzero, then one can divide the ith row of D by the corresponding $S_i(t)$ and obtain the gross *returns* in different states of the world.

- **Portfolio**

  A portfolio is a particular combination of assets in question. To form a portfolio, one needs to know the positions taken in each asset under consideration. The symbol θ_i represents the commitment with respect to the ith asset. Identifying all $\{\theta_i, i = 1, ..., N\}$ specifies the portfolio.

  A positive θ_i implies a long position in that asset, while a negative θ_i implies a short position. If an asset is not included in the portfolio, its corresponding θ_i is zero.

  If a portfolio delivers the same payoff in all states of the world, then its value is known exactly and the portfolio is *riskless*.

Do we have a riskless asset?

A Basic Example of Asset Pricing

- **A Basic Example of Asset Pricing**

  We assume that time consists of "now" and a "next period" and that these two periods are separated by an interval of length Δ (represent a "small" but noninfinitesimal interval).

  We consider a case where the market participant is interested only in three assets:

  1. A risk-free asset such as Treasury bill, whose gross return until next period is $(1 + r\Delta)$. This return is "risk-free", in that it is constant regardless of the realized state of the world.
  2. An *underlying asset*, for example, a stock $S(t)$. We assume that during the small interval Δ, $S(t)$ can assume one of only two possible values. This means a minimum of two states of the world. $S(t)$ is risky because its payoff depends on the states.
  3. A derivative asset, a call option whith premium $C(t)$ and a strike price C_0. The option expires "next" period.

A Basic Example of Asset Pricing (continued)

- **A Basic Example of Asset Pricing**

  This setup is fairly simple. There are three assets ($N = 3$), and two states of the world (K=2). The first asset is risk-free borrowing and lending, the second is the underlying security, and the third is the option.

  We summarize this information in terms of the formal notation discussed earlier. Asset prices will form a vector S_t of only three elements,

$$S_t = \begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} \tag{6}$$

  where $B(t)$ is riskless borrowing or lending, $S(t)$ is a stock, and $C(t)$ is the value of a call option written on this stock.

A Basic Example of Asset Pricing (continued)

- **A Basic Example of Asset Pricing**

  The $B(t)$ is riskless borrowing or lending. Its payoff will be the same, regardless of the state of the world that applies in the "next instant". The $S(t)$ is risky and its value may go either up to $S_1(t+\Delta)$ or go down to $S_2(t+\Delta)$. Finally, the market value of the call option $C(t)$ will change in line with movements in the underlying asset price $S(t)$.

  Payoffs will be grouped in a matrix D_t, as discussed earlier. Thus, D_t will be given by:

$$D_t = \begin{bmatrix} (1+r\Delta)B(t) & (1+r\Delta)B(t) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \tag{7}$$

  where r is the annual riskless rate of return.

A First Glance at the Arbitrage Theorem

- **A First Glance at the Arbitrage Theorem**

  Lets further simplify the notation:

$$B(t) = 1 \quad (8)$$

$$\Delta = 1 \quad (9)$$

THEOREM: Given the S_t, D_t defined in (6) and (7), and given that the two states have positive probabilities of occurrence,

1. if *positive* constants ψ_1, ψ_2 can be found such that

$$\begin{bmatrix} 1 \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ S_1(t+1) & S_2(t+1) \\ C_1(t+1) & C_2(t+1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad (10)$$

2. if there are no arbitrage opportunities, then positive constants ψ_1, ψ_2 satisfying (10) can be found.

A First Glance at the Arbitrage Theorem

- **A First Glance at the Arbitrage Theorem (continued)**

  What do the constants ψ_1, ψ_2 represent? According to the second row of the representation implied by the arbitrage theorem, if a security pays 1 in state 1, and 0 in state 2, then

$$S(t) = (1)\psi_1 \tag{11}$$

  Thus, investors are willing to pay ψ_1 (current) units for an "insurance policy" that offers one unit of account in state 1 and nothing in state 2. Similarly, ψ_2 indicates how much investors would like to pay for an "insurance policy" that pays 1 in state 2 and nothing in state 1.

  Clearly by spending $\psi_1 + \psi_2$, one can guarantee 1 unit of account in the future, regardless of which state is realized. $\psi_i, i = 1, 2$ are called *state prices*.

Relevance of the Arbitrage Theorem

- **A First Glance at the Arbitrage Theorem (continued)**

  The arbitrage theorem provides a very elegant and general method for pricing derivative assets. Consider again

$$\begin{bmatrix} 1 \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ S_1(t+1) & S_2(t+1) \\ C_1(t+1) & C_2(t+1) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{12}$$

  Multiplying the first row of D_t by the vector of ψ_1, ψ_2, we get

$$1 = (1+r)\psi_1 + (1+r)\psi_2 \tag{13}$$

$$\tilde{P}_1 = (1+r)\psi_1 \text{ and } \tilde{P}_2 = (1+r)\psi_2 \tag{14}$$

  Note $0 < \tilde{P}_i \leq 1, \tilde{P}_1 + \tilde{P}_2 = 1$. $\tilde{P}_i$ are called *risk-adjusted* synthetic probabilities.

The Use of Synthetic Probabilities

- **A First Glance at the Arbitrage Theorem (continued)**

  Risk-adjusted probabilities exist if there are no arbitrage opportunities. In other words, if there are no "mispriced assets", we are guaranteed to find positive to find positive constants ψ_1, ψ_2. Multiplying these by the riskless gross return $1 + r$ guarantees the existence of $\tilde{P}_1, \tilde{P}_2$.

  The importance of risk-adjusted probabilities for asset pricing stems from the following: Expectations calculated with them, once discounted by the risk-free rate r, equal the current value of the asset.

  Consider the equality implied by the arbitrage theorem again. Equation (10) implies

$$1 = (1 + r)\psi_1 + (1 + r)\psi_2 \tag{15}$$

The Use of Synthetic Probabilities

- **The Use of Synthetic Probabilities (continued)**

$$S(t) = \psi_1 S_1(t+1) + \psi_2 S_2(t+1) \quad (16)$$

$$C(t) = \psi_1 C_1(t+1) + \psi_2 C_2(t+1) \quad (17)$$

Now multiply the right-hand side of the last two equations by

$$\frac{1+r}{1+r} \quad (18)$$

$$S(t) = \frac{1}{(1+r)}[(1+r)\psi_1 S_1(t+1) + (1+r)\psi_2 S_2(t+1)] \quad (19)$$

$$C(t) = \frac{1}{(1+r)}[(1+r)\psi_1 C_1(t+1) + (1+r)\psi_2 C_2(t+1)] \quad (20)$$

The Use of Synthetic Probabilities

- **The Use of Synthetic Probabilities (continued)**

  But, we can replace $(1+r)\psi_i, i = 1, 2$ with the corresponding $\tilde{P}_i, i = 1, 2$. The two equations become

$$S(t) = \frac{1}{(1+r)}[\tilde{P}_1 S_1(t+1) + \tilde{P}_2 S_2(t+1)] \quad (21)$$

$$C(t) = \frac{1}{(1+r)}[\tilde{P}_1 C_1(t+1) + \tilde{P}_2 C_2(t+1)] \quad (22)$$

  Interpretation: $1/(1+r)$ is a riskless one-period discount factor; the terms in the brackets are expectations calculated using the risk-adjusted probabilities.

  As such, the equalities in (21) and (22) do not represent "true" expected values. Yet as long as there is no arbitrage, these equalities are valid, and they can be used in calculations.

The Use of Synthetic Probabilities

- **The Use of Synthetic Probabilities (continued)**

  We can obtain the "true" expected values by using the "true" probabilities denoted by P_1, P_2:

$$\mathbf{E}^{true}[S(t+1)] = [P_1 S_1(t+1) + P_2 S_2(t+1)] \quad (23)$$

$$\mathbf{E}^{true}[C(t+1)] = [P_1 C_1(t+1) + P_2 C_2(t+1)] \quad (24)$$

  Because these are "risky" assets, when discounted by the risk-free rate, we have following inequalities.

$$S(t) < \frac{1}{(1+r)} \mathbf{E}^{true}[S(t+1)] \quad (25)$$

$$C(t) < \frac{1}{(1+r)} \mathbf{E}^{true}[C(t+1)] \quad (26)$$

  Note: risky assets are negatively correlated with the "market".

The Use of Synthetic Probabilities

- **The Use of Synthetic Probabilities (continued)**

  If no-arbitrage implies the existence of positive constants such as ψ_1, ψ_2, then we can always obtain from these constants the risk-adjusted probabilities $\tilde{P}_1, \tilde{P}_2$ and work with "synthetic" expectations that satisfy

$$S(t) = \frac{1}{(1+r)} \mathbf{E}^{\tilde{P}}[S(t+1)] \tag{27}$$

$$C(t) = \frac{1}{(1+r)} \mathbf{E}^{\tilde{P}}[C(t+1)] \tag{28}$$

  These equations are very convenient to use, and they internalize any risk premiums. Indeed, one does not need to calculate the risk premium if one uses synthetic expectations.

Equalization of Rates of Return

- **Equalization of Rates of Return**

  In the arbitrage-free representation given in (10), divide both sides of the equality by the current price and multiply both sides by $(1+r)$, the gross rate of riskless return.

  Assume nonzero asset prices, we obtain

$$\tilde{P}_1 \frac{S_1(t+1)}{S(t)} + \tilde{P}_2 \frac{S_2(t+1)}{S(t)} = (1+r) \tag{29}$$

$$\tilde{P}_1 \frac{C_1(t+1)}{C(t)} + \tilde{P}_2 \frac{C_2(t+1)}{C(t)} = (1+r) \tag{30}$$

  We notice the gross rates of return of $S(t)$ in state 1 and 2

$$\frac{S_1(t+1)}{S(t)}, \frac{S_2(t+1)}{S(t)} \tag{31}$$

  Under $\tilde{P}_1, \tilde{P}_2$, all expected returns equal the risk-free return r.

The No-Arbitrate Condition

- **The No-Arbitrate Condition**

  Within this simple setup we can also see explicitly the connection between the no-arbitrage condition and the existence of ψ_1, ψ_2. Let the gross returns in states 1 and 2 be given by $R_1(t+1)$ and $R_2(t+1)$ respectively:

$$R_1(t+1) = \frac{S_1(t+1)}{S(t)} \tag{32}$$

$$R_2(t+1) = \frac{S_2(t+1)}{S(t)} = (1+r) \tag{33}$$

  Now write the first two rows of (12) in these new symbols:

$$1 = (1+r)\psi_1 + (1+r)\psi_2$$

$$1 = R_1\psi_1 + R_2\psi_2$$

The No-Arbitrate Condition

- **The No-Arbitrate Condition**

  Subtract the first equation from the second to obtain:

$$0 = ((1+r) - R_1)\psi_1 + ((1+r) - R_2)\psi_2, \tag{34}$$

where we want ψ_1, ψ_2 to be positive. This will be the case, and the above equation will be satisfied if and only if:

$$R_1 < (1+r) < R_2$$

Consider the two examples:

$$(1+r) < R_1 < R_2 \text{ and } R_1 < R_2 < (1+r)$$

which implies, in this simple setting, that there are no arbitrage possibilities.

An Application: Lattice Models

Lattice Models are also called tree models. They are the most common asset pricing methods.

Consider a call option C_t written on the underlying asset S_t. The call option has strike price C_0 and expires at time $T, t < T$. It is known that at expiration, the value of the option is given by

$$C_T = max[S_T - C_0, 0]. \tag{35}$$

We first divide the time interval $(T - t)$ into n smaller intervals, each of size Δ. We choose a "small" Δ, in the sense that the variations of S_t during Δ can be approximated reasonably well by an *up* or *down* movement only. According to this, we hope that for small enough Δ the underlying asset price S_t cannot wander too far from the observed price S_t.

An Application: Lattice Model

Thus, we assume that during Δ the only possible changes in S_t are an *up* movement by $\sigma\sqrt{\Delta}$ or a *down* movement by $-\sigma\sqrt{\Delta}$:

$$S_{t+\Delta} = \begin{cases} S_t + \sigma\sqrt{\Delta} \\ S_t - \sigma\sqrt{\Delta} \end{cases} \tag{36}$$

Clearly, the size of the parameter σ determines how far $S_{t+\Delta}$ can wander during a time interval of length Δ. For that reason it is call the volatility parameter.

The dynamics described by Equation (36) represent a *lattice* or a *binomial tree*. Figure 1 displays these dynamics in the case of *multiplicative* up and down movements.

Suppose now that we are given the (constant) risk-free rate r for the period Δ. Can we determine the risk-adjusted probabilities?

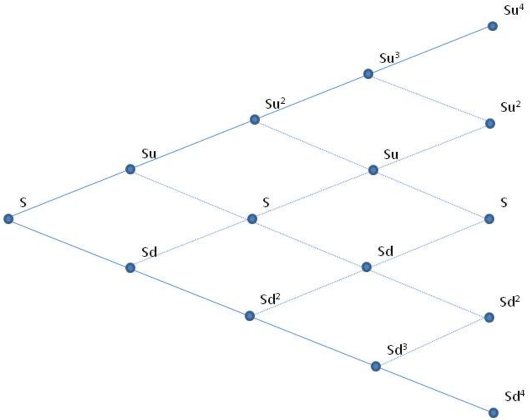

Figure : 1

An Application: Lattice Model (continued)

We know from the arbitrage theorem that the risk-adjusted probabilities $\tilde{P}_{up}$ and $\tilde{P}_{down}$ must satisfy

$$S_t = \frac{1}{1+r}\left[\tilde{P}_{up}(S_t + \sigma\sqrt{\Delta}) + \tilde{P}_{down}(S_t - \sigma\sqrt{\Delta})\right] \quad (37)$$

In this equation, r, S_t, σ and Δ are known. The first three are observed in the markets, while Δ is selected by us. Thus, the only unknown is the $\tilde{P}_{up}$. Once this is done, the $\tilde{P}_{up}$ can be used to calculate the current arbitrage-free value of the call option. In fact the equation

$$C_t = \frac{1}{1+r}\left[\tilde{P}_{up}C^{up}_{t+\Delta} + \tilde{P}_{down}C^{down}_{t+\Delta}\right] \quad (38)$$

"ties" two (arbitrage-free) values of the call option at any time $t + \Delta$ to the (arbitrage-free) value of time t.

An Application: Lattice Model (continued)

We need to know the two values $\tilde{P}_{up}$ and $\tilde{P}_{down}$. Given these, we can calculate the value of the call option C_t at time t.

Figure 2 shows the multiplicative lattice for the option price C_t. The arbitrage-free values of C_t are at this point indeterminate, except for the expiration "nodes". The values of C_t at the expiration using the boundary condition

$$C_t = max[S_T - K, 0]. \tag{39}$$

Once this is done, one can go *backward* using

$$C_t = \frac{1}{1+r}\left[\tilde{P}_{up} C_{t+\Delta}^{up} + \tilde{P}_{down} C_{t+\Delta}^{down}\right] \tag{40}$$

Repeating this several times, one eventually reaches the initial node that gives the current value of the option.

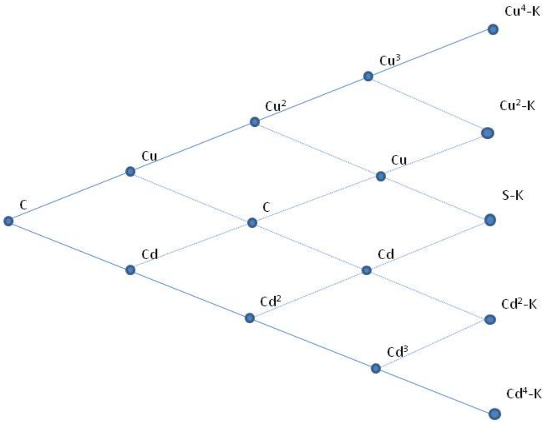

Figure : 2

Payouts and Foreign Currencies

Now we introduce two complications that are more often the case in practical situations. The first is the payment of interim payouts such as dividends and coupons. The second complication is the case of foreign currency denominated assets.

- **The Case with Dividends**

  The simple setup can be modified by adding a dividend equal to d_t percent of $S_{t+\Delta}$.

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B^u_{t+\Delta} & B^d_{t+\Delta} \\ S^u_{t+\Delta} + d_t S^u_{t+\Delta} & S^d_{t+\Delta} + d_t S^d_{t+\Delta} \\ C^u_{t+\Delta} & C^d_{t+\Delta} \end{bmatrix} \begin{bmatrix} \psi^u_t \\ \psi^d_t \end{bmatrix} \tag{41}$$

  where B, S, C denote the savings account, the stock, and a call option, as usual.

The Case with Dividends

- Note two points: the dividends are not lump-sum payment rate has subscript t instead of $t+\Delta$. Second, the dividend payment rate has subscript t instead of $t+\Delta$. The d_t is known as of time t. The simple model in (10) becomes (41).
- With minor modification, we can apply the same steps and obtain two equations:

$$S = \frac{1+d}{1+r}\left[\tilde{P}^u S^u + \tilde{P}^d S^d\right] \tag{42}$$

$$C = \frac{1}{1+r}\left[\tilde{P}^u C^u + \tilde{P}^d C^d\right] \tag{43}$$

where $\tilde{P}$ is the risk-neutral probability. Note that the first equation is now different from the case with no-dividends, but the second equation is the same.

The Case with Dividends (continued)

- According to this, each time an asset has some known percentage payout d during the period Δ, the risk-neutral discounting of the *dividend paying asset* has to be done using the factor $\frac{(1+d)}{(1+r)}$ instead of $\frac{1}{(1+r)}$. Also note that the discounting of the derivative itself did not change.
- Now consider the following transformation:

$$\frac{1+d}{1+r} = \left[\frac{\tilde{P}^u S^u + \tilde{P}^d S^d}{S}\right] \tag{44}$$

which means that the expected return under the risk-free measure is now given by:

$$E^{\tilde{P}}\left[\frac{S_{t+\Delta}}{S_t}\right] = \frac{1+r\Delta}{1+d\Delta}. \tag{45}$$

The Case with Dividends (continued)

Clearly, as a first-order approximation, if d, r are defined over, say, a year, and are *small*:

$$\frac{1 + r\Delta}{1 + d\Delta} \cong 1 + (r - d)\Delta \tag{46}$$

Using this in the previous equation:

$$E^{\tilde{P}}\left[\frac{S_{t+\Delta}}{S_t}\right] \cong 1 + (r - d)\Delta, \tag{47}$$

$$\text{or } E^{\tilde{P}}\left[S_{t+\Delta}\right] \cong S_t + (r - d)S_t\Delta, \tag{48}$$

or, adding a random, unpredictable component, $\sigma S_t \Delta W_{t+\Delta}$:

$$E^{\tilde{P}}\left[S_{t+\Delta}\right] \cong S_t + (r - d)S_t\Delta + \sigma S_t \Delta W_{t+\Delta}. \tag{49}$$

The Case with Dividends (continued)

If we were to let Δ go to zero and switch to continuous time, the *drift* term for dS_t, which represents expected change in the underlying asset's price, will be given by $(r-d)S_t dt$ and the corresponding dynamics can be written as:

$$dS_t = (r-d)S_t dt + \sigma S_t dW_t, \tag{50}$$

where dt represents an infinitesimal time period.

Go over similar steps using the equation (43) for C_t

$$C = \frac{1}{1+r}\left[\tilde{P}^u C^u + \tilde{P}^d C^d\right] \tag{51}$$

We obtain

$$E^{\tilde{P}}\left[\frac{C_{t+\Delta}}{C_t}\right] = 1 + r\Delta. \tag{52}$$

The Case with Foreign Currencies

Suppose we spend e_t units of domestic currency to buy one unit of foreign currency. Thus the e_t is the exchange rate at time t. Assume U.S. dollars (USD) is the domestic currency.

Suppose also that the foreign savings interest rate is known and is given by r^f. The opportunities in investment and the yields of these investments over Δ now be summarized using the following setup:

$$\begin{bmatrix} 1 \\ 1 \\ C(t) \end{bmatrix} = \begin{bmatrix} (1+r) & (1+r) \\ \frac{e^u_{t+\Delta}}{e_t}(1+r^f) & \frac{e^d_{t+\Delta}}{e_t}(1+r^f) \\ C^u_{t+\Delta} & C^d_{t+\Delta} \end{bmatrix} \begin{bmatrix} \psi^u_t \\ \psi^d_t \end{bmatrix} \quad (53)$$

where C_t denotes a call option on price e_t of one unit of foreign currency. The strike price is K.

The Case with Foreign Currencies (continued)

We proceed in a similar fashion as before and obtain the following pricing equations:

$$e = \frac{1 + r^f}{1 + r} \left[\tilde{P}^u e^u + \tilde{P}^d e^d \right] \quad (54)$$

$$C = \frac{1}{1 + r} \left[\tilde{P}^u C^u + \tilde{P}^d C^d \right] \quad (55)$$

Note the first equation is different but the second equation is the same. Thus, each time we deal with a foreign currency denominated asset with payout r^f during Δ, the risk-neutral discounting of the foreign asset has to be done using the factor $(1 + r)/(1 + r^f)$. Note the first-order approximation:

$$\frac{1 + r\Delta}{1 + r^f \Delta} \cong 1 + (r - r^f)\Delta \quad (56)$$

The Case with Foreign Currencies (continued)

The expected rate of return of the e_t and C are different under the probability $\tilde{P}$:

$$E^{\tilde{P}}\left[\frac{e_{t+\Delta}}{e_t}\right] \cong 1 + (r - r^f)\Delta \quad (57)$$

$$E^{\tilde{P}}\left[\frac{C_{t+\Delta}}{C_t}\right] = 1 + r\Delta. \quad (58)$$

According to the last remark, if we were to let Δ go to zero and switch to SDE's. the drift terms for dC_t will be given by $rC_t dt$.

But the drift term for the foreign currency denominated asset, de_t, will now have to be $(r - r^f)e_t dt$.

$$de_t = (r - r^f)e_t dt + \sigma e_t dW_t, \quad (59)$$

Generalization

Up to this point, the setup has been very simple. In general, such simple examples cannot be used to price real-life financial assets.

- **Time Index:** Up to this point we considered discrete time with $t = 1, 2, \ldots$. In continuous-time asset pricing models, this will change. We have to assume that $t \in [0, \infty)$. This way, we need to consider infinitesimal intervals dt.
- **State of World:** In continuous time, there may be *uncountably many* possibilities and a continuum of states of the world. To capture such generalizations, we need to introduce *stochastic differential equations*.
- **Discounting:** If t is continuous, then the discount factor for an interval of length Δ will be given by the *exponential function* $e^{-r\Delta}$. The r becomes the continuously compounded interest rate.

A Methodology of Pricing Assets

The arbitrage theorem provides a powerful methodology for determining fair market values of financial assets in practice. The major steps of this methodology can be summarized as follows:

1. Obtain a model (approximation) to track the dynamics of the underlying asset's price.
2. Calculate how the derivative asset price relates to the price of the underlying asset at expiration or at other boundaries.
3. Obtain risk-adjusted probabilities.
4. Calculate expected payoffs of derivatives at expiration using these risk-adjusted probabilities.
5. Discount this expectation using the risk-free return.

In order to apply this pricing methodology, one needs to be familiar with the following mathematical tools: continuous-time analysis, stochastic calculus, the Girsanov theorem, martingales, stochastic differential equations (SDE).