FE610 Stochastic Calculus for Financial Engineers

Lecture 4. Pricing Derivatives

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Outline

The problem of pricing derivatives is to find a function $F(S_t, t)$ that relates the price of the derivative product to S_t, the price of the underlying asset, and possibly to some other market factors. When the *closed-form* formula is impossible to determine, onc can find numerical ways to describe the dynamics of $F(S_t, t)$.

- One method is to use the notion of *arbitrage* to determine a probability measure under which financial assets behave as *martingales*, once discounted properly. The tools of martingale arithmetic become available, and one can easily calculate arbitrage-free prices, by evaluating the implied expectations. This approach of pricing derivatives is called the *method of equivalent martingale measures*.
- The second pricing method that utilizes arbitrage takes a somewhat more direct approach. One first constructs a risk-free portfolio, and then obtains a *partial differential equation* (PDE) that is implied by the lack of arbitrage opportunities. This PDE is either solved analytically or evaluated numerically.

- *Martingales and Submartingales*

  Suppose at time t one has information summarized by I_t. A random variable X_t that satisfies the equality

$$E^P[X_{t+s}|I_t] = X_t \text{ for all } s > 0, \tag{1}$$

  is called a *martingale with respect to the probability* P. If

$$E^Q[X_{t+s}|I_t] \geq X_t \text{ for all } s > 0, \tag{2}$$

  X_t is called a *submartingale* with respect to probability Q.

  A *martingale* is a model of a fair game where knowledge of past events never helps predict future winnings. In particular, a martingale is a stochastic process for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values.

- *Martingales and Submartingales (continued)*

  According to the discussion in the previous section, asset prices discounted by the risk-free rate will be the risk-adjusted probabilities, but become martingales under the risk-adjusted probabilities. The *fair market values* of the assets under consideration can be obtained by exploiting the martingale equality.

$$X_t = E^{\tilde{P}}[X_{t+s}|I_t] \text{ where } s > 0, \tag{3}$$

$$\text{and } X_{t+s} = \frac{1}{(1+r)^s} S_{t+s}. \tag{4}$$

  Here S_{t+s} and r are the security price and risk-free return, respectively. $\tilde{P}$ is the risk-adjusted probability. According to this, utilization of risk-adjusted probabilities will convert all asset prices into martingales.

It is important to realize that, in finance, the notion of martingale is always associated with two concepts

First, a martingale is always defined with respect to a certain probability. Hence, the discounted price,

$$X_{t+s} = \frac{1}{(1+r)^s} S_{t+s}, \tag{5}$$

is a martingale with respect to the risk-adjusted probability $\tilde{P}$.

Second, note that it is not the S_t that is a martingale, but rather the S_t divided, or *normalized*, by the $(1+r)^s$.

Supposed we divide the S_t by C_t. Would the ratio

$$X^*_{t+s} = \frac{S_{t+s}}{C_{t+s}}, \tag{6}$$

be a martingale with respect to some other probability, say P^*? The answer to this question is positive and is quite useful in pricing interest sensitive derivative instruments.

- **Pricing Functions**

  The unknown of a derivative pricing problem is a *function* $F(S_t, t)$, where S_t is the price of the underlying asset and t is the time. Ideally, the financial analyst will try to obtain a *close-form* formula for $F(S_t, t)$.

  The Black-Scholes formula that gives the price of a call option in terms of the underlying asset and some other relevant parameters is perhaps the bes-known case.

  In cases in which a closed-form formula does not exist, the analyst tries to obtain an equation that governs the *dynamics* of $F(S_t, t)$.

  We will show examples of how to determine such $F(S_t, t)$. The discussion is intended to introduce new mathematical tools and concepts that have common use in pricing derivative products.

  1. Forwards $F(S_t, t)$
  2. Options $C_t = F(S_t, t)$

- **Forwards:**

  In particular, we consider a forward contract with the following provisions:

  - At some future date T, where

    $$t < T, \tag{7}$$

    F dollars will be paid for one unit of gold.
  - The contract is signed at time t, but no payment changes hands until time T.

- Hence, we have a contract that imposes an obligation on both counterparties - the one that delivers the gold, and the one that accepts the delivery.

  How can one determine a function $F(S_t, t)$ that gives the fair market value of such a contract at time t in terms of the underlying parameters? We use an arbitrage argument.

1). Suppose one buys one unit of physical gold at time t for S_t dollars using funds borrowed at the continuously compounding risk-free rate r_t. The r_t is assumed to be fixed during the contract period.

Let the insurance and storage costs per time unit be c dollars and let them be paid at time T.

The total cost of *holding* this gold during a period of length $T - t$ will be given by

$$e^{r_t(T-t)} S_t + (T - t)c, \tag{8}$$

where the first term is the principal and interest to be returned to the bank at time T, and the second represents total storage and insurance costs paid at time T.

This is one method of securing one unit of physical gold at time T. One borrows the necessary funds, buys the underlying commodity, and stores it until time T.

2). The forward contract is another way of obtaining a unit of gold at time T. One signs a contract now for delivery of one unit of gold at time T, with the understanding that all payments will be made at the expiration.

Hence, the outcomes of the two sets of transactions are identical. This means that they must cost the same; otherwise, there will be arbitrage opportunities.

Mathematically, this gives the equality

$$F(S_t, t) = e^{r_t(T-t)} S_t + (T-t)c, \tag{9}$$

Thus we used the possibility of exploiting any arbitrage opportunities and obtained an equality that expresses the price of a forward contract $F(S_t, t)$ as a function of S_t and other parameters.

** The function $F(S_t, t)$ is *linear* in S_t. Later we will derive a *non-linear* formula - Black-Sholes formula.

Boundary Conditions

Suppose we want to express formally the notion that the "expiration date gets nearer". To do this, we use the concept of limits. We let

$$t \to T. \tag{10}$$

Note that as this happends

$$\lim_{t \to T} e^{r_t(T-t)} = 1. \tag{11}$$

Apply the limit to the left-hand side of the expression in (9), we obtain

$$S_T = F(S_T, T). \tag{12}$$

It means, at expiration, the cash price of the underlying asset and the price of the forward contract will be equal.

- **Options**

  Determine the pricing function $F(S_t, t)$ for nonlinear assets is not as easy as in the case of forward contracts. Here we prepare the groundwork for further mathematical modeling.

  Suppose C_t is a call option written on the stock S_t. Let r be the constant risk-free rate. K is the strike price, and T, $t < T$, is the expiration date. Then the price of the call option can be expressed as

$$C_t = F(S_t, t). \tag{13}$$

  Under simplifying conditions, the S_t will be the only source of randomness affecting the option's price. Hence, unpredictable movements in S_t can be offset by opposite positions taken simultaneously in C_t. This property imposes some conditions on the way $F(S_t, t)$ can change over time once the time path of S_t is given.

- *Pricing Functions - Options (continued)*

Property of the the pricing function $F(S_t, t)$.

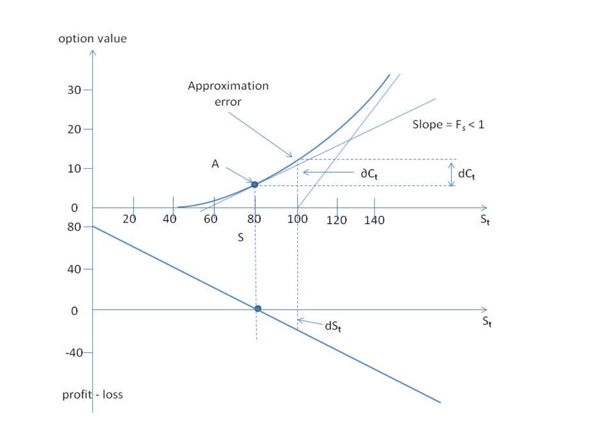

Figure : 2 - **A unit of the underlying asset S_t is borrowed and sold at price S.**

- *Pricing Functions - Options (continued)*
  - The first panel of Figure 1 displays the price $F(S_t, t)$ of a call option written on S_t. The lower part of this figure displays a payoff diagram for a short position in S_t.
  - Suppose, originally, the underlying asset's price is S. That is, initially we are at point A on the $F(S_t, t)$ curve. If the stock price increase by dS_t, the short position will lose exactly the amount dS_t. But the option position gains.
  - According to Figure 1, when S_t increases by dS_t, the price of the call option will increase only by dC_t; this latter change is smaller because the slope of the curve is less than one, i.e.,

$$dC_t < dS_t. \tag{14}$$

Hence, if we owned one call option and sold one stock, a price increase equal to dS_t would lead to a net loss. But it suggests that with careful adjustments, such losses could be eliminated.

- *Pricing Functions - Options (continued)*
  - Consider the slope of the tangent to $F(S_t, t)$ at point A. This slope is given by
$$\frac{\partial F(S_t, t)}{\partial S_t} = F_s. \quad (15)$$
  - Now, suppose we are short by not one, but by F_s units of the underlying stock. Then, as S_t increases by dS_t, the total loss on the short position will be $F_s dS_t$. But according to Figure 1, this amount is very close to dC_t. It is indicated by ∂C_t.
  - Clearly, if dS_t is a small incremental change, then the ∂C_t will be a very good approximation of the actual change dC_t. As a result, the gain in the option position will (approximately) offset the loss in the short position.
  - Thus, incremental movements in $F(S_t, t)$ and S_t should be related by some equation, and it can be used in finding a closed-form formula for $F(S_t, t)$.

- *Pricing Functions - Options (continued)*
  - We can write

$$d[F_S S_t] + d[F(S_t, t)] = g(t). \tag{16}$$

where $g(t)$ is a completely predictable function of time t.

[**DEFINITION** Offsetting changes in C_t by taking the opposite position in F_s units of the underlying asset is called *delta hedging*. Such a portfolio is *delta neutral*, and the parameter F_s is called the *delta*.]

  - It is important to realize that when dS_t is "large",

$$\partial C_t \cong dC_t. \tag{17}$$

the approximation will fail. With an extreme movement, the "hedge" may be less satisfactory. The assumption of *continuous* time plays implicitly a fundamental role.

- *Pricing Functions - Options (continued)*

Property of the the pricing function $F(S_t, t)$.

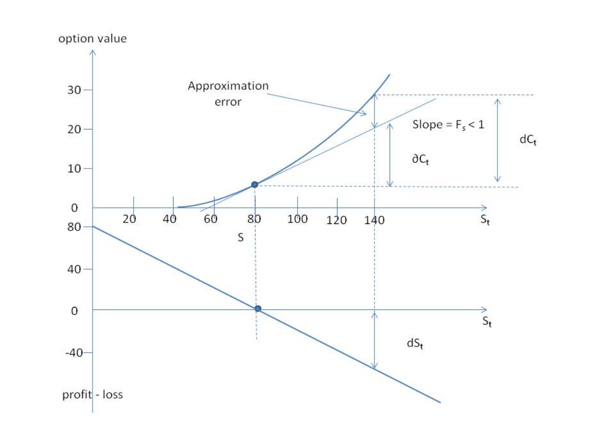

Figure : 2 - **A unit of the underlying asset S_t is borrowed and sold at price S.**

Application: Another Pricing Method

- **Application: Another Pricing Method**

  We use the discussion of the previous section to summarize the pricing method that uses partial differential equations.

  1. Assume that an analyst observes the current price of a derivative product $F(S_t, t)$ and the underlying asset price S_t in real time. Suppose the analyst would like to calculate the change in the derivative asset's price $dF(S_t, t)$, given a change in the price of the underlying asset dS_t.
  2. Remember that the concept of differentiation is a tool that one can use to approximate small *changes* in a function. In this particular case, we indeed have a function $F(\cdot)$ that depends on S_t, t. Thus, we would write

$$dF(S_t, t) = F_s dS_t + F_t dt, \tag{18}$$

  where the F_i are partial derivatives,

- **Application: Another Pricing Method (continued)**

  and we have the partial derivatives as

$$F_s = \frac{\partial F}{\partial S_t} \text{ for all } F_t = \frac{\partial F}{\partial t} \tag{19}$$

  and where $dF(S_t, t)$ denotes the total changes.

- Equation (18) equation can be used once the partial derivatives F_s, F_t are evaluated numerically. This, on the other hand, requires that the functional form of $F(S_t, t)$ be known.
- Once the stochastic version of Equation (18) is determined, one can complete "program" for valuing a derivative asset in the following way. Using delta-hedging and risk-free portfolios, one can obtain additional relationships among $dF(S_t, t), dS_t$, and dt. One would then obtain a relationship that ties only the partial derivatives of $F(\cdot)$ to each other. if one has enough boundary conditions, and if a closed-form solution exists.

Example

Suppose we know that the partial derivative of $F(x)$ with respect to $x \in [0, X]$ is known constant, b:

$$F_x = b \tag{20}$$

This equation is a trivial PDE. It is an expression involving a partial derivative of $F(x)$. Using this PDE, we can tell the *form* of the function $F(x)$? The answer is yes. Only linear relationships have a property such as (20). Thus $F(x)$ must be given by

$$F(x) = a + bx. \tag{21}$$

The *form* of $F(x)$ is pinned down. However, the parameter a is still unknown. It is found by using the so-called "boundary conditions". For example, if we know that the boundary $x = X, F(X) = 10$, the a can be determined by $a = 10 - bX$.

The Problem

Financial market data are not *deterministic*. Hence, $F(S_t, t), S_t$ and possibly the risk-free rate r_t are all continuous-time *stochastic processes*. We can not apply the standard calculus tools.

- **A First Look at Ito's Lemma**

- In standard calculus, variables under consideration are deterministic. Hence, to get a relation such as

$$dF(t) = F_s dS_t + F_r dr_t + F_t dt, \tag{22}$$

- The change in $F(\cdot)$ is given by the relation on the right-hand side of (22). But according to the rules of calculus, this equation holds exactly only during infinitesimal intervals. In *finite* time intervals, Eq. (22) will hold only as an approximation. Consider again the univariate Taylor series expansion.

- *The Problem (continued)*

  Let $f(x)$ be an infinitely differentiable function of $x \in R$. On can then write the Taylor series expansion of $f(x)$ around $x_0 \in R$ as

$$f(x) = f(x_0) + f_x(x_0)(x - x_0) + \frac{1}{2} f_{xx}(x_0)(x - x_0)^2 + \frac{1}{3} f_{xxx}(x_0)(x - x_0)^3 + ... = \sum_{i=0}^{\infty} \frac{1}{i!} f^i(x_0)(x - x_0)^i, \tag{23}$$

  where $f^i(x_0)$ is the ith-order partial derivative of $f(x)$ with respect to x, evaluated at x_0.

  We can reinterpret $df(x)$ using the approximation

$$df(x) \cong f(x) - f(x_0) \text{ and } dx \text{ as } dx \cong (x - x_0) \tag{24}$$

- *The Problem (continued)*

  Thus, an expression such as

$$dF(t) = F_s dS_t + F_r dr_t + F_t dt \tag{25}$$

  depends on the assumption that the terms $(dt)^2, (dS_t)^2$, and $(dr_t)^2$, and those of higher order, are "small" enough that they can be omitted from a multivariate Taylor series expansion. Because of such an approximation, higher powers of the differentials dS_t, dt, or dr_t do not show up on the right-hand side of (25).

  Now, dt is a small deterministic change in t. So to say that $(dt)^2, (dt)^3, \ldots$ are "small" with respect to dt is an internally consistent statements. However, the same argument cannot be used for $(dS_t)^2$, and possibly $(dr_t)^2$.

- First, $(dS_t)^2$ and $(dr_t)^2$ are random during small intervals, and they have *nonzero* variances during dt.

- *The Problem (continued)*

  This posses a problem: On one hand, we want to use continuous-time random processes with nonzero variables during dt. So, we use positive numbers for the average values of $(dS_t)^2$ and $(dr_t)^2$. But under these conditions, it would be inconsistent to call $(dS_t)^2$ and $(dr_t)^2$ "small" with respect to dt, and equate them to zero.

- In a stochastic environment with a continuous flow of randomness, we can write the relevant total differentials as:

$$dF(t) = F_s dS_t + F_r dr_t + F_t dt + \frac{1}{2} F_{ss} dS_t^2 + \frac{1}{2} F_{rr} dr_t^2 + F_{sr} dS_t dr_t. \quad (26)$$

We want to learn how to exploit the chain rule in a stochastic environment and understand what a differential means in such a setting.