FE610 Stochastic Calculus for Financial Engineers

Lecture 6. Martingales and Martingale Representations

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Outline

Continuous Time Martingales

Using different information sets, one can conceivably generate different "forecasts" of a process $\{S_t\}$. These forecasts are expressed using conditional expectations. In particular,

$$E_t[S_T] = E[S_T|I_t], t < T, \tag{1}$$

is the formal way of denoting the forecast of a future value, S_T of S_t, using the information available as of time t. $E_u[S_T], u < t$, would denote the forecast of the same variable using a smaller information set as of or earlier than time u.

DEFINITION: We say that a process $\{S_t, t \in [0, \infty]\}$ is a martingale with respect to the family of information sets I_t and with respect to the probability P, if, for all $t > 0$,

Continuous Time Martingales (continued)

1. S_t is known, give I_t. (S_t is I_t-adapted.)
2. Unconditional "forecasts" are finite:

$$E|S_t| < \infty. \tag{2}$$

3. And if

$$E_t[S_T] = S_t, \text{for all } t < T, \tag{3}$$

with probability 1. That is, the best forecast of unobserved future values is the last observation on S_t.

Here, all expectations $E[\cdot]$, $E_t[\cdot]$ are assumed to be taken with respect to the probability P.

[According to this definition, martingales are random variables whose future variations are completely unpredictable given the current information set.]

The Use of Martingales in Asset Pricing

Now, we know that stock prices or bond prices are not completely unpredictable. The price of a discount bond is expected to *increase* over time. In general, the same is true for stock prices. They are expected to increase on average.

Hence, if B_t represents the price of a discount bond maturing at time $T, t < T$,

$$B_t < E_t[B_u], t < u < T. \tag{4}$$

Clearly, the price of a discount bond is not a martingale.

- Similarly, in general, a risky stock S_t will have a positive expected return will not be a martingale. For a small interval Δ, we can write

$$E_t[S_{t+\Delta} - S_t] \cong \mu\Delta, \tag{5}$$

where μ is a positive rate of expected return.

The Use of Martingales in Asset Pricing (continued)

- It turns out that although most financial assets are not martingales, one can convert them into martingales. For example, one can find a probability distribution $\tilde{P}$ such that bond or stock prices discounted by the risk-free rate become martingales. If this is done, equalities such as

$$E_t^{\tilde{P}}[e^{-ru}B_{t+u}] = B_t, t < u < T - t. \tag{6}$$

for bonds, or

$$E_t^{\tilde{P}}[e^{-ru}S_{t+u}] = S_t, 0 < u. \tag{7}$$

for stock prices, can be very useful in pricing derivative securities.

- Two ways of converting submartingales into martingales:

* The first method should be obvious. We can subtract an *expected trend* from $e^{-rt}S_t$ or $e^{-rt}B_t$. This would make the deviations around the trend completely unpredictable.

The Use of Martingales in Asset Pricing (continued)

** The second method is to transform its *probability distribution*. That is, if one had

$$E_t^P[e^{-ru}S_{t+u}] > S_t, 0 < u. \tag{8}$$

where $E_t^P[\cdot]$ is the conditional expectation calculated using a probability distribution P, we may try to find an "equivalent" probability $\tilde{P}$, such that the new expectations satisfy

$$E_t^{\tilde{P}}[e^{-ru}S_{t+u}] = S_t, 0 < u. \tag{9}$$

and the $e^{-rt}S_t$ becomes a martingale. The probability distributions that convert equation above into equality are called *equivalent martingale measures* - A transformation based on *Girsanov Theorem* is more promising than the Doob-Meyer decompositions.

Relevance of Martingales in Stochastic Modeling

Let X_t represent an asset price that has the martingale property with respect to the filtration $\{I_t\}$ and with respect to the probability $\tilde{P}$,

$$E^{\tilde{P}}[X_{t+\Delta}|I_t] = X_t, \tag{10}$$

where $\Delta > 0$ represents a small time interval. What type of trajectories would such an X_t have in continuous time?

To answer this question, first define the *martingale difference* ΔX_t,

$$\Delta X_t = X_{t+\Delta} - X_t, \tag{11}$$

and then note that since X_t is a martingale,

$$E^{\tilde{P}}[\Delta X_t|I_t] = 0. \tag{12}$$

Relevance of Martingales in Stochastic Modeling (continued)

- As mentioned earlier, this equality implies that increments of a martingale should be totally unpredictable, no matter how small the time interval Δ is. But, since we are working with continuous time, we can indeed consider very small Δ's. Martingales should then display very irregular trajectories. In fact, X_t should not display any trends discernible by inspection, even during infinitesimally small time intervals.
- Such irregular trajectories can occur in two different ways. They can be *continuous*, or they can dsiplay *jumps*. The former leads to *continuous martingales*, whereas the latter are called *right continuous martingales*.
- Suppose one is dealing with a continuous martingale X_t that also has finite second moment $E[X_t^2] < \infty$ for all$t > 0$. Such a process is called a *continuous square integrable martingale*. It is very close to Brownian motion.

Example

Here, we will construct a martingale using two independent Poisson processes observed during "small intervals" Δ.

Suppose financial markets are influenced by "good" and "bad" news. We ignore the content of the news, but retain the information on whether it is "good" or "bad".

The N_t^G and N_t^B denote the total number of instances of "good" and "bad" news, respectively,until time t. We assume further that the way news arrives in financial markets is totally unrelated to past data, and that the "good" and "bad" news are independent.

Finally, during a small interval Δ, at most one instance of good news or bad news can occur, and the probability of this occurrence is the same for both types of news. Thus, the probabilities of incremental changes $\Delta N^G, \Delta N^B$ during Δ is assumed to be given approximately by

Example

$$P(\Delta N_t^G = 1) = P(\Delta N_t^B = 1) \cong \lambda\Delta. \tag{13}$$

Then the variable M_t, defined by

$$M_t = N_t^G - N_t^B, \tag{14}$$

will be a martingale. Note that the increments of M_t over small intervals Δ will be given by

$$\Delta M_t = \Delta N_t^G - \Delta N_t^B. \tag{15}$$

Apply the conditional expectation operator:

$$E_t[\Delta M_t] = E_t[\Delta N_t^G] - E_t[\Delta N_t^B]. \tag{16}$$

But, approximately,

$$E_t[\Delta N_t^G] \cong 0 \cdot (1 - \lambda\Delta) + 1 \cdot \lambda\Delta \cong \lambda\Delta, \tag{17}$$

Example

Similarly, we get

$$E_t[\Delta N_t^B] \cong 0 \cdot (1 - \lambda\Delta) + 1 \cdot \lambda\Delta \cong \lambda\Delta, \tag{18}$$

This means that

$$E_t[\Delta M_t] \cong \lambda\Delta - \lambda\Delta = 0. \tag{19}$$

Hence, increments in M_t are unpredictable given the family I_t. However, if we assume that "good" news can occur with a slightly greater probability than "bad" news,

$$P(\Delta N_t^G = 1) = \lambda^G \Delta > P(\Delta N_t^B = 1) \cong \lambda^B \Delta. \tag{20}$$

Then M_t will cease to be martingale (but a submartingale), since

$$E_t[\Delta M_t] \cong \lambda^G \Delta - \lambda^B \Delta > 0. \tag{21}$$

Properties of Martingale Trajectories

Assume that $\{X_t\}$ represents a trajectory of a continuous square integrable martingale. Pick a time interval $[0, T]$ and consider the times $\{t_i\}$:
$t_0 = 0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$.
We define the variation of the trajectory as

$$V^1 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}| \tag{22}$$

Heuristically, V^1 can be interpreted as the length of the trajectory followed by X_t during the interval $[0, T]$.
The *quadratic* variation is given by

$$V^2 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2 \tag{23}$$

Properties of Martingale Trajectories (continued)

One can similarly define *higher-order* variations. For example, the fourth-order variation is defined as:

$$V^4 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^4 \tag{24}$$

- Remember that we want X_t to be continuous and to have a nonzero variance. This means two things:

  First, as the partitioning of the interval $[0, T]$ gets finer and finer, "consecutive" X_t's get nearer and nearer, for any $\epsilon > 0$

$$P(|X_{t_i} - X_{t_{i-1}}| > \epsilon) \to 0, \tag{25}$$

  Second, as the partitions get finer and finer, we still want

$$P(|\sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2 > 0) = 1. \tag{26}$$

Properties of Martingale Trajectories (continued)

Now, consider some properties of V^1 and V^2.

- First, note that even though X_t is a continuous martingale, and X_{t_i} approaches $X_{t_{i-1}}$ as the subinterval $[t_i - t_{i-1}]$ becomes smaller and smaller, this does not mean that V^1 also approaches zero. The reader may find this surprising. After all, V^1 is made of the sum of such incremental changes:

$$V^1 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}| \tag{27}$$

Surprisingly, the opposite is true. As $[0, T]$ is partitioned into finer and finer subintervals, changes in X_t get smaller. But, at the same time, the number of terms in the sum defining V^1 increases. It turns out that in the case of a continuous-time martingale, the second effect dominates and the V^1 goes toward infinity.

Properties of Martingale Trajectories (continued)

This can be shown heuristically as follows. We have

$$\sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2 < [\max_i |X_{t_i} - X_{t_{i-1}}|] \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}| \tag{28}$$

because the right-hand side is obtained by factoring out the "largest" $|X_{t_i} - X_{t_{i-1}}|$. This means that

$$V^2 < [\max_i |X_{t_i} - X_{t_{i-1}}|] V^1 \tag{29}$$

As $t_i \to t_{i-1}$ for all i the continuity of the martingale implies that "consecutive" X_{t_i}'s will get very near each other. At the limit,

$$\max_i |X_{t_i} - X_{t_{i-1}}| \to 0. \tag{30}$$

It implies that we must have $V^1 \to \infty$.

Properties of Martingale Trajectories (continued)

Now consider the same property for higher-order variations. Consider V^4 and apply the same "trick" as before:

$$V^4 < [\max_i |X_{t_i} - X_{t_{i-1}}|^2] V^2 \tag{31}$$

As long as V^2 converges to a well-defined random variable, the right-hand side of (31) will go to zero. At the limit,

$$\max_i |X_{t_i} - X_{t_{i-1}}|^2 \to 0. \tag{32}$$

It implies that we must have $V^4 \to 0$. Here we summarize the three properties of the trajectories:

1. The variation V^1 will converge to infinity in some probabilistic sense and the continuous martingale will behave very irregularly.

Properties of Martingale Trajectories (continued)

2 The quadratic variation V^2 will converge to some well-defined random variable.

3 All higher-order variations will vanish in some probabilistic sense.

These properties have important implications. First, we see that V^1 is not a very useful quantity to use in the calculus of continuous square integrable martingale, whereas the V^2 can be used in a meaningful way. Second, higher-order variations can be ignored if one is certain that the underlying process is a *continuous* martingale.

Furthermore, since the Riemann-Stieltjes integral uses the equivalent of V^1 in deterministic calculus and considers finer and finer partitions of interval under consideration. In stochastic environments such limits do not converge. Instead, stochastic calculus is forced to use V^2.

Example

Example: Brownian Motion

- Suppose X_t represents a continuous process whose increments are normally distributed. Such a process is called a (generalized) Brownian motion. We observe a value of X_t for each t. At every instant, the infinitesimal changes in X_t is denoted by dX_t. Incremental changes in X_t are assumed to be independent across time.

  Under these conditions, if Δ is small interval, the increments ΔX_t during Δ will have a normal distribution with mean $\mu\Delta$ and variance $\sigma^2\Delta$. This means

$$\Delta X_t \sim N(\mu\Delta, \sigma^2\Delta). \tag{33}$$

  The fact that increments are uncorrelated can be expressed as

$$E[(\Delta X_u - \mu\Delta)(\Delta X_t - \mu\Delta)] = 0, u \neq t. \tag{34}$$

Example

Example: Brownian Motion

- Leaving aside formal aspects of defining such a process X_t here we ask a simple question: is X_t a martingale?

  The process X_t is the "accumulation" of infinitesimal increments over time, that is,

$$X_{t+T} = X_0 + \int_0^{t+T} dX_u. \tag{35}$$

  Assuming that the integral is well defined, we can calculate the relevant expectations. Consider the expectation take with respect to the probability distribution given in (33), and given the information on X_t observed up to time t:

$$E[X_{t+T}] = E_t[X_t + \int_t^{t+T} dX_u]. \tag{36}$$

Example

Examples of Martingales (continued)

But at time t, future values of ΔX_{t+T} are predictable because all changes during small intervals Δ have expectation equal to $\mu\Delta$. This means

$$E_t[\int_t^{t+T} dX_u] = \mu T. \tag{37}$$

So,

$$E[X_{t+T}] = X_t + \mu T. \tag{38}$$

Clearly, $\{X_t\}$ is not a martingale with respect to the distribution in Eq. (33) and with respect to the information on current and past X_t. But, this last result gives a clue to how to generate a martingale with $\{X_t\}$. Consider the process:

$$Z_t = X_t - \mu t. \tag{39}$$

Example

Examples of Martingales (continued)

It is easy to show that Z_t is a martingale:

$$E[Z_{t+T}] = E[X_{t+T} - \mu(t+T)] \tag{40}$$

$$= E[X_t + (X_{t+T} - X_t)] - \mu(t+T)]. \tag{41}$$

which means

$$E[Z_{t+T}] = X_t + E[(X_{t+T} - X_t)] - \mu(t+T). \tag{42}$$

But the expectation on the right-hand side is equal to μT, as shown in Eq. (38). This means

$$E[Z_{t+T}] = X_t - \mu t = Z_t \tag{43}$$

That is , Z_t is a martingale. Hence, we were able to transform X_t into a martingale by subtracting a deterministic function.

Martingale Representations

In this section, we formalize these special cases and discuss the so-called Doob-Meyer decomposition.

Suppose a trader observes at times t_i, $t_0 < t_1 < t_2 < ... < t_{n-1} < t_k = T$, the price of a financial asset S_t.

If the intervals between the times t_{i-1} and t_i are very small, and if the market is "liquid", the price of the asset is likely to exhibit at most one uptick or one downtick during a typical $t_i - t_{i-1}$. We formalize this by saying that at each instant t_i, there are only two possibilities for S_{t_i} to change:

$$\Delta S_{t_i} = \begin{cases} 1 & \text{with probability p} \\ -1 & \text{with probability (1-p)} \end{cases} \tag{44}$$

It is assumed that these changes are independent of each other. Further, if $p = 1/2$, then the expected value of ΔS_{t_i} will equal to zero, otherwise nonzero.

Martingale Representations (continued)

Given that a typical object of interest is a *sample path*, or trajectory, of price changes, we first need to construct a set made of all possible paths. This space is called a sample space. Its elements are made of sequences of $+1$'s and -1's. For example, a typical sample path can be

$$\{\Delta S_{t_i} = -1, ..., \Delta S_{t_k} = +1\}, \tag{45}$$

Since k is finite, given an initial point S_{t_0} we can easily determine the trajectory followed by the asset price by adding incremental changes. This way we can construct the set of all possible trajectories, i.e., the sample space.

For example, the particular sequence of $\Delta S*$ that begins with $+1$ at time t_0 and alternates until time t_k,

$$\Delta S^* = \{\Delta S_{t_1} = +1, \Delta S_{t_2} = -1, ..., \Delta S_{t_k} = -1\}, \tag{46}$$

Martingale Representations (continued)

- We will have the probability (assuming k is even)

$$P(S^*) = p^{k/2}(1-p)^{k/2}. \tag{47}$$

  Since k is finite, there are a finite number of possible trajectories in the sample space, and we can assign a probability to every one of these trajectories.

- Another assumption that simplifies this task is the independence of successive price changes. this way, the probability of the whole trajectory can be obtained by simply multiplying the probabilities associated with each incremental change. One can easily obtain the level of the asset price from subsequent changes, given the opening price S_{t_0}:

$$S_{t_t} = S_{t_0} + \sum_{i=1}^{k}(S_{t_i} - S_{t_{i-1}}). \tag{48}$$

Martingale Representations (continued)

- The highest possible value for S_{t_k} is $S_{t_0} + k$. This value will result if all incremental changes $\Delta S_{t_k}, i = 1, ..., k$ are made of $+1$'s. The probability of this outcome is

$$P(S_{t_k} = S_{t_0} + k) = p^k. \tag{49}$$

  Similarly, the lowest possible value of S_{t_k} is $S_{t_0} - k$. The probability of this is given by

$$P(S_{t_k} = S_{t_0} - k) = (1 - p)^k. \tag{50}$$

- In general, the price would be somewhere between these two extremes. Of the k incremental changes observed, m would be made of $+1$'s and $k - m$ made of -1's, with $m \leq k$. The S_{t_k} will assume the value

$$S_{t_k} = S_{t_0} + m - (k - m). \tag{51}$$

  Note: several trajectories may result in the same value.

Martingale Representations (continued)

- Adding the probabilities associated with all these combinations, we obtain

$$P(S_{t_k} = S_{t_0} + 2m - k) = C_k^{k-m} p^m (1-p)^{k-m}, \tag{52}$$

$$C_k^{k-m} = \frac{k!}{m!(k-m)!}. \tag{53}$$

- This probability is given by the *binomial distribution*. As $k \to \infty$, this distribution converges to normal distribution. Consider the expectation under the probabilities in (53).

$$E^p[(S_{t_k} | S_{t_0}, \Delta S_{t_1}, ..., \Delta S_{t_{k-1}}] = S_{t_{k-1}} + [(+1)p + (-1)(1-p)], \tag{54}$$

  where the second term on the right-hand side is the expectation of ΔS_{t_k}, the unknown increments given the information at time $I_{t_{k-1}}$. Clearly, if $p = 1/2$, this term is zero, and we have a martingale.

Martingale Representations (continued)

- The $\{S_{t_k}\}$ will be martingale with respect to the information set generated by past price changes and with respect to this particular probability distribution

$$E^p[(S_{t_k}|S_{t_0}, \Delta S_{t_1}, ..., \Delta S_{t_{k-1}}] = S_{t_{k-1}} \tag{55}$$

- If $p \neq 1/2$, the $\{S_{t_k}\}$ will cease to be a martingale with respect to $\{I_{t_k}\}$. However, the centered process $\{Z_{t_k}\}$, defined by

$$Z_{t_k} = [S_{t_0} + (1-2p)] + \sum_{i=1}^{k} [\Delta S_{t_i} + (1-2p)] \tag{56}$$

or

$$Z_{t_k} = S_{t_k} + (1-2p)(k+1) \tag{57}$$

will again be a martingale with respect to I_{t_k}

Doob-Meyer Decomposition

- Consider the case where the probability of an uptick at any time t_i is somewhat greater than the probability of a downtick for a particular asset, so that we expect a general upward trend in observed trajectories: $1 > p > 1/2$.

  Then, as shown earlier,

$$E^P[S_{t_k}|S_{t_0}, S_{t_1}, ..., S_{t_{k-1}}] = S_{t_{k-1}} - (1 - 2p), \tag{58}$$

  which means,

$$E^P[S_{t_k}|S_{t_0}, S_{t_1}, ..., S_{t_{k-1}}] > S_{t_{k-1}}, \tag{59}$$

  since $2p > 1$. This implies that $\{S_{t_k}\}$ is a *submartingale*. Now, as shown earlier, we can write

$$S_{t_{k-1}} = -(1 - 2p)(k + 1) + Z_{t_k}, \tag{60}$$

  where Z_{t_k} is a martingale.

The General Case

- The decomposition of an upward-trending submartingale into a deterministic trend and a martingale component was done for a process observed at a finite number of points during a continuous interval.
- The Doob-Meyer theorem provides the answer to this question. We state the theorem without proof.

  Let $\{I_t\}$ be the family of information sets discussed above.

  THEOREM: If $X_t, 0 \leq t \leq \infty$ is a right-continuous submartingale with respect to the family $\{I_t\}$ and if $E[X_t] < \infty$ for all t, then X_t admits

$$X_t = M_t + A_t, \tag{61}$$

  where M_t is a right-continuous martingale with respect to probability P, and A_t is an increasing process measurable with respect to I_t.

The Use of Doob-Meyer Decomposition

- The Doob-Meyer decomposition shows that even if continuously observed asset prices contain occasional jumps and trend upwards at the same time, then we can convert them into martingales by subtracting a process observed at time t.
- We assume again that time $t \in [0, T]$ is continuous. The value of a call option C_t written on the underlying asset S_t will be given by the function

$$C_T = max[S_T - K, 0] \tag{62}$$

at expiration date T. At an earlier time $t, t < T$, the exact value of C_T is unknown. But we can calculate a forecast of it using the information I_t available at time t,

$$E^P[C_T|I_t] = E^P[max[S_T - K, 0]|I_t] \tag{63}$$

The Use of Doob-Meyer Decomposition (continued)

- Given this forecast, one may be tempted to ask if the fair market value C_t will equal a properly discounted value of $E^P[max[S_T - K, 0]|I_t]$.
- For example, suppose we use the (constant) risk-free interest rate r to discount $E^P[max[S_T - K, 0]|I_t]$, to write

$$C_T = e^{-r(T-t)} E^P[max[S_T - K, 0]|I_t]. \tag{64}$$

Would this equation give the fair value C_t of the call option? The answer depends on whether or not $e^{-rt}C_t$ is a martingale with respect to the pair I_t, P. If it is, we have

$$E^P[e^{-rt}C_T|I_t] = e^{-rt}C_t, t < T, \tag{65}$$

Then $e^{-rt}C_t$ will be a martingale.

But can we expect $e^{-rt}S_t$ to be martingale under the true probability P?

The Use of Doob-Meyer Decomposition (continued)

- As discussed earlier, under the assumption that investors are risk-averse, for a typical risky security we have

$$C_T = E^P[e^{-r(T-t)}S_T|S_t] > S_t. \tag{66}$$

  That is $e^{-rt}S_t$ will be a submartingale.
- But, according to Doob-Meyer decomposition, we can decompose the $e^{-rt}S_t$ to obtain

$$e^{-rt}S_t = A_t + Z_t, t < T, \tag{67}$$

  where A_t is an increasing I_t measurable random variable, and Z_t is a martingale with respect to the information I_t.
- However, in practice, it is more convenient and significantly easier to convert asset prices into martingales, not by subtracting their drift, but instead by changing the underlying probability distribution P.

The First Stochastic Integral

- Let $H_{t_{i-1}}$ be any random variable adapted to $I_{t_{i-1}}$. Let Z_t be any martingale with respect to I_t and to some probability measure P. Then the process defined by

$$M_{t_k} = M_{t_0} + \sum_{i=1}^{k} H_{t_{i-1}}[Z_{t_i} - Z_{t_{i-1}}] \tag{68}$$

  will also be a martingale with respect to I_t.
- Z_t is a martingale and has unpredictable increments. The fact that $H_{t_{i-1}}$ is $I_{t_{i-1}}$-adapted means $H_{t_{i-1}}$ are "constants" given $I_{t_{i-1}}$. Then, increments in Z_{t_i} will be uncorrelated with $H_{t_{i-1}}$ as well. Using these observations, we can calculate

$$E_{t_0}[M_{t_k}] = M_{t_0} + E_{t_0}[\sum_{i=1}^{k} E_{t_{i-1}}[H_{t_{i-1}}(Z_{t_i} - Z_{t_{i-1}})]]. \tag{69}$$

The First Stochastic Integral (continued)

But increments in Z_{t_i} are unpredictable as of time t_{i-1}. Also, $H_{t_{i-1}}$ is I_t-adapted. This means we can move the $E_{t_{i-1}}[\cdot]$ operator "inside" to get

$$H_{t_{i-1}} E_{t_{i-1}}[Z_{t_i} - Z_{t_{i-1}}] = 0. \quad (70)$$

This implies

$$E_{t_0}[M_{t_k}] = M_{t_0}. \quad (71)$$

M_t thus has the martingale property.

It turns out that M_t defined this way is the first example of a *stochastic integral*. The questions is whether we can obtain a similar results when $\sup_i[t_i - t_{i-1}]$ goes to zero. Using some analogy, we can obtain an expression

$$M_t = M_0 + \int_0^t H_u dZ_u. \quad (72)$$

Application to Finance: Trading Gains

- We consider a decision maker who invests in both a riskless and a risky security at trading time t_i:

$$0 = t_0 < ... < t_i < ... < t_n = T. \tag{73}$$

  Let $\alpha_{t_{i-1}}$ and $\beta_{t_{i-1}}$ be the number of shares of riskless and risky securities held by the investor right before time t_i trading begins. Clearly, these random variables will be I_{t_i}-adapted. α_{t_i} and β_{t_i} are the nonrandom initial holdings.
- Suppose we now consider trading strategies that are *self-financing*. These are strategies where time t_i investments are financed solely from the proceeds of time t_{i-1} holdings. That is, they satisfy

$$\alpha_{t_{i-1}} B_{t_i} + \beta_{t_{i-1}} S_{t_i} = \alpha_{t_i} B_{t_i} + \beta_{t_i} S_{t_i}, \tag{74}$$

  where $i = 1, 2, ..., n$.

Application to Finance: Trading Gains (continued)

- According to this strategy, the investor can sell his holdings at time t_i for an amount equal to the left-hand side of the equation, and with all of these proceeds purchase α_{t_i} and β_{t_i} units of riskless and risky securities. We can now substitute recursively for the left-hand side using Eq (74) for $t_{i-1}, t_{i-2}, \ldots$, and using the definitions

$$B_{t_i} = B_{t_{i-1}} + [B_{t_i} - B_{t_{i-1}}] \tag{75}$$

$$S_{t_i} = S_{t_{i-1}} + [S_{t_i} - S_{t_{i-1}}] \tag{76}$$

We obtain

$$\alpha_{t_0} B_{t_0} + \beta_{t_0} S_{t_0} + \sum_{j=1}^{i-1} [\alpha_{t_j}[B_{t_i} - B_{t_{i-1}}] + \beta_{t_j}[S_{t_i} - S_{t_{i-1}}]] = \alpha_{t_i} B_{t_i} + \beta_{t_i} S_{t_i} \tag{77}$$

where the RHS is the wealth of the decision maker after time t_i trading.