FE610 Stochastic Calculus for Financial Engineers

Lecture 9. Integration in Stochastic Environment - The Ito Integral

Steve Yang

Stevens Institute of Technology

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Outline

The Ito Integral and SDEs

Obtaining a formal definition of the Ito integral will make the notion of a stochastic differential equation more precise. Once the integral $\int_0^1 \sigma(S_u, u) dWu$
is defined in some precise sense, the one could integrate both sides of the SDE:

$$S_{t+h} - S_t = \int_t^{t+h} a(S_u, u) du + \int_t^{t+h} \sigma(S_u, u) dW_u, \quad (1)$$

If h is small, $a(S_u, u)$ and $\sigma(S_u, u)$ may not change very much during $u \in [t, t+h]$, especially if they are smooth functions of S_u and u. Then we could rewrite this equation as:

$$\Delta S_t \cong a(S_u, u) \int_t^{t+h} du + \sigma(S_u, u) \int_t^{t+h} dW_u, \quad (2)$$

The Ito Integral and SDEs (continued)

The SDE representation is an *approximation* for at least two reasons.

- First, the $E_t[S_{t+h} - S_t]$ was set equal to a first-order Taylor series approximation with respect to h:

$$E_t[S_{t+h} - S_t] = a(S_u, u)h. \tag{3}$$

- Second, the $a(S_u, u), \sigma(S_u, u), u \in [t, t+h]$ were approximated by their value at $u = t$.

All these imply that when we write

$$dS_t = a(S_u, u)dt + \sigma(S_t, t)dW_t, \tag{4}$$

we in fact mean that in the integeral equation

$$\int_t^{t+h} S_u = \int_t^{t+h} a(S_u, u)du + \int_t^{t+h} \sigma(S_u, u)dW_u, \tag{5}$$

The Ito Integral and SDEs (continued)

As $h \to 0$, the second integeral in (5) is defined as

$$\int_t^{t+h} \sigma(S_u, u) dW_u \cong \sigma(S_t, t) dW_t. \tag{6}$$

That is, the diffusion terms of the SDEs are in fact Ito integrals approximated during infinitesimal time intervals.

- Considering the SDE written over intervals h:

$$\sum_{k=1}^{n-1}[S_k - S_{k-1}] = \sum_{k=1}^{n-1}[a(S_{k-1}, k)h + \sum_{k=1}^{n-1} \sigma(S_{k-1}, k)[\Delta W_k] \tag{7}$$

- Can we define a limit similar to the standard integeral:

$$\int_0^T dS_u = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n-1}[a(S_{k-1}, k)h + \sum_{k=1}^{n-1} \sigma(S_{k-1}, k)[\Delta W_k] \right\} \tag{8}$$

The Ito Integral and SDEs (continued)

- The first term on the right-hand side of (8) does not contain any random terms once information in time k.
- The second term on the right-hand side of (8) contains random variables even after I_{k-1} is revealed. The sum

$$\sum_{k=1}^{n-1} \sigma(S_{k-1}, k)[\Delta W_k] \tag{9}$$

is an integral with respect to a random variable. It turns out that, under some conditions, it is possible to define a stochastic integral as the limit in mean square of the random sum. This integral is a *random variable* that has a variance that goes to zero as n increases toward infinity.

$$\lim_{n\to\infty} E\left[\sum_{k=1}^{n-1}[\sigma(S_{k-1}, k)[W_k - W_{k-1}] + \int_0^T \sigma(S_u, u)[dW_u]\right]^2 = 0 \tag{10}$$

The Ito Integral and SDEs (continued)

- **DEFINITION:** Consider the finite interval approximation of the stochastic differential equation

$$S_k - S_{k-1} = a(S_{k-1}, k)h + \sigma(S_{k-1}, k)[W_k - W_{k-1}], k = 1, 2, ..., n.$$

where $[W_k - W_{k-1}]$ is a standard Wiener process with zero mean and variance h. We let

(1) the $\sigma(S_t, t)$ be nonanticipative, in the sense that they are independent of the future; and

(2) the random variable $\sigma(S_t, t)$ be "non-explosive":

$$E\left[\int_0^T \sigma(S_t, t)^2 dt\right] < \infty. \quad (11)$$

Then the Ito integral

$$\int_0^T \sigma(S_u, u) dW_u \quad (12)$$

The Ito Integral and SDEs (continued)

The Ito Integral is the mean square limit,

$$\lim_{n\to\infty} E\left[\sum_{k=1}^{n-1}[\sigma(S_{k-1},k)[W_k - W_{k-1}] + \int_0^T \sigma(S_u,u)[dW_u]\right]^2 = 0 \quad (13)$$

According to this definition, as the number of intervals goes to infinity and the length of each interval becomes infinitesimal, the finite sum will approach the random variable represented by the Ito integral. Clearly, the definition makes sense only if such a limiting random variable exists.

To summarize, we see three major differences between deterministic and stochastic integrations:

- **First**, the notion of limit used in stochastic integration is different.
- **Second**, the Ito integral is defined for nonanticipative functions only.
- **Third**, while integrals in standard calculus are defined using the actual "paths" followed by functions, stochastic integrals are defined within *stochastic equivalence*.

An Ito Integral Example

The Ito integral is the mean square limit of a certain finite sum. Thus, in order for the Ito integral to exist, some appropriate sums must converge.

Suppose one has to evaluate the integral

$$\int_0^T x_t dx_t, \tag{14}$$

where it is known that $x_0 = 0$.

We would first partition the interval $[0, T]$ into n smaller subintervals all of size h

$$t_0 = 0 < t_1 < \ldots < t_n = T, \tag{15}$$

where $T = nh$ and $t_{i+1} - t_i = h$. If x_t is a Wiener process, the standard calculus approach cannot be used.

An Ito Integral Example (continued)

- First of all, we must define V_n as

$$V_n = \sum_{i=1}^{n-1} x_{t_i}[x_{t_{i+1}} - x_{t_i}]. \tag{16}$$

  In other words, x_t has to be evaluated at time t_i instead of at t_{i+1}, because otherwise these terms will fail to be *nonanticipating*. The $x_{t_{i+1}}$ will be unknown as of time t_i, and will be correlated with the increments $[x_{t_{i+1}} - x_{t_i}]$.
- Second, V_n is now a random variable and simple limits cannot be taken. In taking the limit of V_n, one has to use a probabilistic approach. We have to determine a limiting random variable V such that

$$\lim_{n \to 0} E[\sum_{i=1}^{n-1} x_{t_i}[x_{t_{i+1}} - x_{t_i}] - V]^2 = 0. \tag{17}$$

An Ito Integral Example (continued)

We begin by noting that for any a and b we have

$$(a+b)^2 = a^2 + b^2 + 2ab,$$
$$\text{or } ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2].$$

and letting $a = x_{t_i}$ and $b = \Delta x_{t_{i+1}}$ gives

$$V_n = \frac{1}{2}\sum_{i=0}^{n-1}[(x_{t_i} + \Delta x_{t_{i+1}})^2 - x_{t_i}^2 - x_{t_{i+1}}^2]. \quad (18)$$

But $x_{t_i} + \Delta x_{t_{i+1}} = x_{t_{i+1}}$ which gives:

$$V_n = \frac{1}{2}\left[\sum_{i=0}^{n-1} x_{t_{i+1}}^2 - \sum_{i=0}^{n-1} x_{t_i}^2 - \sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2\right]. \quad (19)$$

Now simplify the summation items and note $x_0 = 0$.

An Ito Integral Example (continued)

We can obtain the following:

$$V_n = \frac{1}{2}\left[x_T^2 - \sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2\right]. \tag{20}$$

Note that x_T is independent of n, and hence the mean square limit of V_n will be determined by the mean square limit of the term $\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2$. We look for the Z in

$$\lim_{n\to\infty} E\left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - Z\right]^2. \tag{21}$$

First, we consider a candidate:

$$E\left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2\right] \tag{22}$$

An Ito Integral Example (continued)

Expectation (22) will be a good candidate for Z. Taking expectation in a straightforward way.

$$E\left[\sum_{i=0}^{n-1}\Delta x_{t_{i+1}}^2\right] = \sum_{i=0}^{n-1}E[\Delta x_{t_{i+1}}^2] = \sum_{i=0}^{n-1}(t_{i+1}-t_i) = T. \tag{23}$$

Now using this as a candidate for Z, we can calculate the expectation

$$E\left[\sum_{i=0}^{n-1}\Delta x_{t_{i+1}}^2 - T\right]^2 =$$

$$E\left[\sum_{i=0}^{n-1}\Delta x_{t_{i+1}}^4 + 2\sum_{i=0}^{n-1}\sum_{j<i}^{n-1}[\Delta x_{t_{i+1}}^2][\Delta x_{t_{j+1}}^2] + T^2 - 2T\sum_{i=0}^{n-1}\Delta x_{t_{i+1}}^2\right] \tag{24}$$

Now we consider the components of the above equation individually.

An Ito Integral Example (continued)

We know that Wiener process increments are independent,

$$E[\Delta x_{t_{i+1}}^2 \Delta x_{t_{j+1}}^2] = (t_{i+1} - t_i)(t_{j+1} - t_j) \tag{25}$$

and

$$E[\Delta x_{t_{i+1}}^4] = 3((t_{i+1} - t_i)^2, \tag{26}$$

we obtain

$$E\left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - T\right]^2 =$$

$$\sum_{i=0}^{n-1} 3((t_{i+1} - t_i)^2 + 2\sum_{i=0}^{n-1}\sum_{j<i}^{n-1}(t_{i+1} - t_i)(t_{j+1} - t_j)$$

$$+T^2 - 2T\sum_{i=0}^{n-1}(t_{i+1} - t_i) \tag{27}$$

An Ito Integral Example (continued)

Now we use the fact that $(t_{i+1} - t_i) = h$, for all i, since all intervals are the same size. We have the following:

$$E\left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - T\right]^2 = 2nh^2 + n(n-1)h^2 - n^2h^2 = 2Th, \qquad (28)$$

This implies that as $n \to \infty$, the size of the intervals will go to zero, and

$$\lim_{n\to\infty} E\left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - Z\right]^2 = \lim_{n\to\infty} 2hT = 0. \qquad (29)$$

Thus, the mean square limit of $\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2$ is T.

$$\lim_{n\to\infty} E[V_n]^2 = \frac{1}{2}\left[x_T^2 - T\right]. \qquad (30)$$

An Ito Integral Example (continued)

The Ito integral is given by

$$\int_0^T x_t dx_t = \frac{1}{2}\left[x_T^2 - T\right] \tag{31}$$

Assume that x_t is a Wiener process and consider

$$\int_0^T (dx_t)^2 \tag{32}$$

which can be interpreted as the sum of squared increments in x_t. If this integral exists in the Ito sense, then by definition

$$\lim_{n\to\infty} E\left[\sum_{i=0}^{n-1} \Delta x_{t_{i+1}}^2 - \int_0^T (dx_t)^2\right]^2 = 0. \tag{33}$$

An Ito Integral Example (continued)

We then obtain a result that may seem a bit "unusual" to one who is used to working with standard calculus:

$$\int_0^T (dx_t)^2 = \int_0^T dt, \tag{34}$$

where the equality holds in the mean square sense. It is in this sense that if W_t represents a Wiener process, for infinitesimal dt, one can write:

$$(\mathbf{dW_n})^2 = \mathbf{dt}. \tag{35}$$

In fact, in all practical calculations dealing with stochastic calculus, it is a common practice to replace the terms involving dW_t^2 by dt. The preceding discussion traces the logic behind this procedure. The equality should be interpreted in the sense of mean squared convergence.

The Ito Integral is a Martingale

Models that describe the dynamic behavior of asset prices contain innovation terms that represent unpredictable news. As a result, an integral of the form

$$\int_t^{t+\Delta} \sigma_u dW_u \qquad (36)$$

is a sum of unpredictable disturbances that affect asset prices during an interval of length Δ.

Now, if each increment is unpredictable given the information set at time t, the sum of these increments should also be unpredictable. This makes the integral a martingale difference:

$$E_t \left[\int_t^{t+\Delta} \sigma_u dW_u \right] = 0 \qquad (37)$$

Properties of the Ito Integral (continued)

- The integral

$$\int_0^t \sigma_u dW_u \tag{38}$$

becomes a martingale

$$E_{k-1}\left[\int_0^t \sigma_u dW_u\right] = \int_0^s \sigma_u dW_u, 0 < s < t. \tag{39}$$

Hence, the existence of unpredictable innovation terms in equations describing the dynamics of asset prices coincides well with the martingale property of the Ito integral.

The condition that ensures this martingale property is the one that requires σ_t be nonanticipative given the information set I_t.

Properties of the Ito Integral (continued)

- **Existence**

  One can ask the question: when does the Ito integral of a general random function $f(S_t, t)$,

$$\int_0^t f(S_u, u) dS_u, \tag{40}$$

  where $\{S_t\}$ is given before. It turns out that if the function $f(\cdot)$ is continuous, and if it is nonanticipating, this integral exists. In other words, the finite sums

$$\sum_{i=0}^{n-1} f(S_{t_i}, t_i)[S_{t_{i+1}} - S_{t_i}] \tag{41}$$

  converge in mean square to "some" random variable that we call the Ito "Integral".

Properties of the Ito Integral (continued)

- **Correlation Properties**

The martingale property gives the first moment of the integral of a monanticipating $f(\cdot)$ with respect to a Wiener process

$$E\left[\int_0^T f(W_t, t)dW_t\right] = 0, \tag{42}$$

where W_t is a Wiener process. The second moments are given by the variance and covariances

$$E\left[\int_0^t f(W_u, u)dW_u \int_0^t g(W_u, u)dW_u\right] = \int_0^t E[f(W_u, u)g(W_u, u)]du \tag{43}$$

$$\text{and } E\left[\int_0^t f(W_u, u)dW_u\right]^2 = E\left[\int_0^t f(W_u, u)^2 du\right]. \tag{44}$$

Properties of the Ito Integral (continued)

- **Addition**

  In particular, the integral of the sum of two (random) :

$$\int_0^T [f(S_t,t)d + g(S_t,t)]dS_t = \int_0^T f(S_t,t)dS_t + \int_0^T g(S_t,t)dS_t \quad (45)$$

- **Integral with Respect to Jump Processes**

  Suppose a process M_t is a martingale that exhibits finite jumps only and has no Wiener component. Trajectories of such an M_t will exhibit occasional jumps, but otherwise will be very smooth. Then one could define a V_n,

$$V_n = \sum_{i=0}^{n-1} f(M_{t_i})[M_{t_{i+1}} - M_{t_i}], \quad (46)$$

This V_n will converge, and the variation of the process M_t will be finite with probability one. Under these conditions, we say that V_n converges pathwise.