FE610 Stochastic Calculus for Financial Engineers

Lecture 10. Ito's Lemma

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Outline

Ito's Lemma

- The stochastic version of the chain rule is known as Ito's Lemma.
- Let S_t be a continuous time process which depends on the Wiener process W_t. Suppose we are given a function of S_t denoted by $F(S_t, t)$, and suppose we would like to calculate the change in $F(\cdot)$ when dt amount of time passes.
- Clearly, passing time would influence the $F(S_t, t)$ in two different ways.

  First, there is a direct influence through the t variable in $F(S_t, t)$.

  Second, as time passes, one obtains new information about W_t and observes a new increment, dS_t. This will also make $F(\cdot)$ change.

Ito's Lemma (continued)

- The sum of these two effects is represented by the stochastic differential $dF(S_t, t)$ and is given by the stochastic equivalent of the chain rule.

$$\frac{dF(S_t, t)}{dt} = F_s \frac{dS_t}{dt} + F_t. \tag{1}$$

- We again partition the time interval $[0, T]$ into n equal pieces, each with length h. We apply the Taylor series formula to $F(S_k, k)$, $k = 1, 2, ...,$ where the S_k is assumed to obey

$$\Delta S_k = a_k h + \sigma_k \Delta W_k. \tag{2}$$

First, fix k. Given the information set I_{k-1}, S_{k-1} is a known number.

Next, apply Taylor formula to expand $F(S_k, k)$ around S_{k-1} and $k - 1$.

Ito's Lemma (continued)

Using Taylor expansion to approximate $F(S_k, k)$:

$$\begin{aligned} F(S_k, k) &= F(S_{k-1}, k-1) + F_s[S_k - S_{k-1}] + F_t[h] \\ &+ \frac{1}{2}F_{ss}[S_k - S_{k-1}]^2 + \frac{1}{2}F_{tt}[h]^2 + F_{st}[h(S_k - S_{k-1})] + R, \end{aligned} \tag{3}$$

where the partials $F_s, F_{ss}, F_{tt}, F_{st}$ are all evaluated at $S_{k-1}, k-1$. R represents the remaining terms.

- Transpose $F(S_{k-1}, k-1)$ and relabel the increments, and substitute these into (3):

$$F(S_k, k) - F(S_{k-1}, k-1) = \Delta F(k) \tag{4}$$

$$S_k - S_{k-1} = \Delta S_k. \tag{5}$$

and

$$\begin{aligned} \Delta F(S_k, k) &= F_s \Delta S_k + F_t[h] + \frac{1}{2}F_{ss}[\Delta S_k]^2 \\ &+ \frac{1}{2}F_{tt}[h]^2 + F_{st}[h(\Delta S_k)] + R, \end{aligned} \tag{6}$$

Ito's Lemma (continued)

- But we know that the dynamics of S_t are governed by Eq. (2), and we can substitute the right-hand side of this for ΔS_k in the Taylor series expansion:

$$\Delta F(S_k, k) = F_s[a_k h + \sigma_k \Delta W_k] + F_t[h] + \frac{1}{2} F_{ss}[a_k h + \sigma_k \Delta W_k]^2 + \frac{1}{2} F_{tt}[h]^2 + F_{st}[h][a_k h + \sigma_k \Delta W_k] + R, \quad (7)$$

- The first-order effects are the effects of time, represented by $F_t[h]$, and the effects of change in the underlying asset's price, $F_s[a_k h + \sigma_k \Delta W_k]$.
- The second-order effects are those changes that are represented by squared terms and by cross products. Higher-order terms are grouped in the remainder R.
- In order to obtain a chain rule in stochastic environments, the terms will be classified as negligible and nonnegligible.

The Notion of "Size" in Stochastic Calculus

- In stochastic settings, the time variable t is still deterministic. So, with respect to the time variable, the same criterion of smallness as in deterministic calculus can be applied.
- On the other hand, the same rationale cannot be used for a stochastic differential such as dS_t^2. We will use

$$dW_t^2 = dt. \tag{8}$$

Hence, terms involving dS_t^2 are likely to have sizes of order dt, which was considered as nonnegligible.
CONVENTION: Given a function $g(\Delta W_k, h)$ dependent on the increments of the Wiener process W_t, and on the time increment, if the ratio

$$\lim_{h \to 0} \frac{g(\Delta W_k, h)}{h} = 0. \tag{9}$$

then we consider $g(\Delta W_k, h)$ as negligible.

The First-Order Terms

Here, the terms that contain h or ΔS_k are clearly first-order increments that are not negligible. As $F_s[a_k h + \Delta W_k]$ or $F_t h$ are divided by h, and h is made smaller and smaller, there terms do not vanish.

$$\lim_{n\to\infty} \frac{F_s a_k h}{h} = F_s a_k \lim_{n\to\infty} \frac{F_t h}{h} = F_t \tag{10}$$

are clearly independent of$h$, and do not vanish as h gets smaller.

On the other hand, we know the ratio

$$\lim_{n\to\infty} \frac{F_s \Delta W_k}{h} \tag{11}$$

gets larger (in a probabilistic sense) as h becomes smaller, since the term ΔW_k is of the order $h^{1/2}$.

The Second-Order Terms

Divide the second-order terms on the right-hand side by h and consider the ratio

$$\frac{F_{tt}h^2}{2h}. \tag{12}$$

This term remains proportional to h, since in the numerator we have an increment that depends on h^2.

Next, consider the second-order term that depends on $[\Delta S_k]^2$,

$$\lim_{n\to\infty} \frac{1}{2}F_{ss}\left[\frac{a_k^2 h^2}{h} + \frac{\sigma_k \Delta W_k}{h}\right]^2 \tag{13}$$

$$\cong \frac{1}{2}F_{ss}\sigma_k^2. \tag{14}$$

The difference between the two sides has a variance that will tend to zero as $h \to 0$.

The Cross-Product and Higher-Order Terms

- Consider the following cross-product term (divide by h):

$$\frac{F_{st}[h][a_k h + \sigma_k \Delta W_k]}{h} = F_{st}[a_k h + \sigma_k \Delta W_k]. \qquad (15)$$

This another way of saying that the Wiener process has continuous sample paths.

Higher-Order Terms

The right-hand side depends on ΔW_k. As $h \to 0$, ΔW_k goes to zero.

All the terms in the remainder R contain powers of h and of ΔW_k greater than 2. According to the convention adopted earlier, if the unpredictable shocks are of "normal" type - i.e., there are no "rare events" - powers of ΔW_k greater than two will be negligible.

The Ito Formula

ITO's LEMMA: Let $F(S_t, t)$ be a twice-differentiable function of t and of the random process S_t:

$$dS_t = a_t dt + \sigma_t dW_t, t \geq 0, \tag{16}$$

with well-behaved drift and diffusion parameters, a_t, σ_t. Then we have

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 dt \tag{17}$$

or, after substituting for dS_t using the relevant SDE,

$$dF_t = \left[\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 \right] + \frac{\partial F}{\partial S_t} \sigma_t dW_t, \tag{18}$$

where the equality holds in the mean square sense.

Uses of Ito's Lemma

- The Ito's Lemma provides a tool for obtaining stochastic differentials for functions of random processes. For example, we want to know what happens to the price of an option if the underlying asset's price changes.
- Letting $F(S_t, t)$ be the option price, and S_t the underlying asset's price, we can write

$$dF_t(S_t, t) = F_s dS_t + F_t dt + \frac{1}{2} F_{ss} \sigma_t^2 dt, \tag{19}$$

  If one has an exact formula for $F(S_t, t)$, one can then take the partial derivative explicitly and replace them in the foregoing formula to get the stochastic differential, $dF(S_t, t)$.
- The second use of Ito's Lemma is quite different. Ito's Lemma is useful in evaluating Ito integrals.

Ito's Formula as a Chain Rule

- Consider a function of the standard Wiener process W_t given by

$$F(W_t, t) = W_t^2. \tag{20}$$

Remember that W_t has a drift parameter 0 and a diffusion parameter 1. Applying the Ito formula to this function,

$$dF_t = \frac{1}{2}[2dt] + 2W_t dW_t \tag{21}$$

or

$$dF_t = dt + 2W_t dW_t \tag{22}$$

Note that Ito's formula results, in this particular case, in an SDE that has $a(I_t, t) = 1$ and $\sigma(I_t, t) = 2W_t$.
- Hence, the drift is constant and the diffusion depends on the information set I_t.

Ito's Formula as a Chain Rule - Example

- Lets apply Ito's formula to the function

$$F(W_t, t) = 3 + t + e^{W_t}. \tag{23}$$

We obtain

$$dF_t = dt + e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt. \tag{24}$$

Grouping,

$$dF_t = \left[1 + \frac{1}{2} e^{W_t}\right] dt + e^{W_t} dW_t \tag{25}$$

In this case, we obtain a SDE for $F(S_t, t)$ with I_t-dependent drift and diffusion terms:

$$a(I_t, t) = \left[1 + \frac{1}{2} e^{W_t}\right] dt \text{ and } \sigma(I_t, t) = e^{W_t} \tag{26}$$

Ito's Formula as an Integration Tool

- Suppose one needs to evaluate the following Ito integral

$$\int_0^t W_s dW_s. \tag{27}$$

We define

$$F(W_t, t) = \frac{1}{2} W_t^2, \tag{28}$$

and apply the Ito formula to $F(W_t, t)$:

$$dF_t = 0 + W_t dW_t + \frac{1}{2} dt \tag{29}$$

This is an SDE with drift $1/2$ and diffusion W_t. Writing the corresponding integral equation,

$$F(W_t, t) = \int_0^t W_s dW_s + \frac{1}{2} \int_0^t ds, \tag{30}$$

Ito's Formula as an Integration Tool (continued)

- Take the second integral to the right-hand side, and apply the definition of $F(W_t, t)$:

$$\frac{1}{2}W_t^2 = \int_0^t W_s dW_s + \frac{1}{2}t. \tag{31}$$

Rearranging terms, we obtain the desired result

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t. \tag{32}$$

It is important to summarize how Ito's formula was exploited to evaluate Ito integrals.

(1) We guessed a form for the function $F(W_t, t)$.
(2) Ito's Lemma was used to obtain the SDE for $F(S_t, t)$.
(3) We applied the integral operator to both sides of this new SDE, and obtained an integral equation.
(4) Rearrange the integral equation gave us the desired result.

Integral Form of Ito's Formula

- As repeatedly mentioned, stochastic differentials are simply shorthand for Ito integrals over small time intervals. One can thus write the Ito formula in integral form.

$$F(S_t, t) = F(S_0, 0) + \int_0^t F_s dS_s + \int_0^t [F_u + \frac{1}{2} F_{ss} \sigma_u^2] du \quad (33)$$

where use has been made of the equality

$$\int_0^t dF_u = F(S_t, t) - F(S_0, 0) \quad (34)$$

We can use the version of the Ito formula shown above in order to obtain another characterization.

$$\int_0^t F_s dS_u = [F(S_t, t) - F(S_0, 0)] - \int_0^t [F_u + \frac{1}{2} F_{ss} \sigma_u^2] du \quad (35)$$

Ito's Formula and Jumps

- Suppose we observe a process S_t, which is believed to follow the SDE

$$dS_t = a_t dt + \sigma_t dW_t + dJ_t, t \geq 0, \tag{36}$$

where dW_t is a standard Wiener process. The new term dJ_t represents possible unanticipated jumps. This jump component has zero mean during a finite interval h:

$$E[\Delta J_t] = 0. \tag{37}$$

This assumption is not restrictive, as any predictable part of the jumps may be included in the drift component a_t.
We assume that between jumps, J_t remains constant. At jump times $\tau_j, j = 1, 2, ...,$ it varies by some discrete and random amount. We assume that there are k possible types of jumps, with sizes $\{a_i, i = 1, 2, ..., k\}$.

Ito's Formula and Jumps (continued)

- The jumps occur at a rate λ_t that may depend on the latest observed S_t. Once a jump occurs, the jump type is selected randomly and independently. The probability that a jump of size a_i will occur is given by p_i.

$$\Delta J_t = \Delta N_t - \left[\lambda_t h \left(\sum_{i=1}^{k} a_i p_i \right) \right] \tag{38}$$

  where N_t is a process that represents the sum of all jumps up to time t. The term $\sum_{i=1}^{k} a_i p_i$ is the expected size of a jump, whereas $\lambda_t h$ represents, loosely speaking, the probability that a jump will occur. There are subtracted from ΔN_t to make ΔJ_t unpredictable.
- Under these conditions, the drift coefficient a_t can be seen as representing the sum of two separate drifts (the Wiener and the Jump processes).

Ito's Formula and Jumps (continued)

- We can, therefore, write the drift as:

$$a_t = \alpha_t + \lambda_t \left(\sum_{i=1}^{k} a_i p_i \right), \text{ and } S_t^- = \lim_{s \to t} S_s, s < t. \quad (39)$$

where α_t is a drift of the stochastic process in S_t.

- The occurrence of a jump is a random event. And the size of the jump is also random. Under these conditions, the Ito formula is given by:

$$dF(S_t, t) = \left[F_t + \lambda_t \sum_{i=1}^{k} (F(S_t + a_i, t) - F(S_t, t)) p_i + \frac{1}{2} F_{ss} \sigma^2 \right] dt$$

$$+ F_s dS_t + [F(S_t, t) - F(S_t^-, t)]$$

$$\lambda_t \left[\sum_{i=1}^{k} (F(S_t + a_i, t) - F(S_t, t)) p_i \right] dt.$$