FE610 Stochastic Calculus for Financial Engineers

Lecture 10. The Dynamics of Derivative Prices - Stochastic Differential Equations

Steve Yang

Stevens Institute of Technology

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Outline

1. Stochastic Differential Equation
2. Verification of Solutions to SDEs
3. Major Models of SDEs

A Geometric Description of Paths Implied by SDEs

- Consider the stochastic differential equation

$$dS_t = a(S_t, t)dt + \sigma(S_t, t)dW_t, t \in [0, \infty), \tag{1}$$

  where the drift and diffusion parameters depend on the level of observed asset price S_t and (possibly) on t. These parameters are themselves random variables. Conditional on available information, they "become" constant.
- The $a(S_t, t)$ and $\sigma(S_t, t)$ parameters are assumed to satisfy the conditions

$$P\left(\int_0^t |a(S_u, u)|du < \infty\right) = 1$$

and

$$P\left(\int_0^t |\sigma(S_u, u)|du < \infty\right) = 1 \tag{2}$$

A Geometric Description of Paths Implied by SDEs (continued)

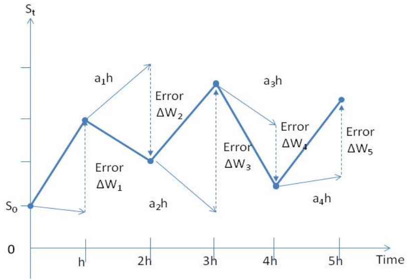

Figure : This geometric derivation emphasizes that the trajectories of S_t are likely to be very erratic when h becomes infinitesimal.

Types of Solutions

- The first type of solution to an SDE is similar to the case of ordinary differential equations. Given the drift and diffusion parameters and the random innovation term dW_t, we determine a random process S_t paths of which satisfy the SDE:

$$dS_t = a(S_t, t)dt + \sigma(S_t, t)dW_t, t \in [0, \infty), \quad (3)$$

Clearly, such a solution S_t will depend on time t, and on the past and contemraneous values of the random variable W_t, as the underlying integral equation illustrates:

$$S_t = S_0 + \int_0^t a(S_u, u)du + \int_0^t \sigma(S_u, u)dW_u, t > 0 \quad (4)$$

When W_t on *the right-hand side is given exogenously* and S_t is then determined, we obtain the so-called **strong solution** of the SDE.

Types of Solutions (continued)

- The second solution concept is specific to stochastic differential equations. It is called *weak solution*. In the weak solution, one determines the process $\tilde{S}_t$,

$$\tilde{S}_t = f(t, \tilde{W}_t), \tag{5}$$

  where $\tilde{W}_t$ is a Wiener process whose distribution is determined *simultaneously* with $\tilde{S}_t$.
- The idea of a weak solution is that given that solving SDEs involves finding random variables that satisfy

$$\int_0^t dS_u = \int_0^t a(S_u, u)du + \int_0^t \sigma(S_u, u)dW_u, t > 0 \tag{6}$$

  One can argue that finding an $\tilde{S}_t$ and a $\tilde{W}_t$ such that the pair $\{\tilde{S}_t, \tilde{W}_t\}$ satisfies this equation is also a type of solution.

Types of Solutions (continued)

- The weak solution is in contrast with strong solutions where one does not solve for the W_t, but considers it another given of the problem.
- If we look at the form of distribution, the density functions of dW_t and $d\tilde{W}_t$ are given by the same formula. In this sense, there is no difference between the two random errors.
- The strong solution calculates an S_t that satisfies the SDE with dW_t *given*. That is, in order to obtain the strong solution S_t, we need to know the family I_t.
- The weak solution $\tilde{S}_t$, on the other hand, is not calculated using the process that generates the information set I_t. Instead, it is found along with some process $\tilde{W}_t$. The process $\tilde{W}_t$ could generate some other information set H_t. The corresponding $\tilde{S}_t$ will not necessarily be I-adapted. But $\tilde{W}_t$ will still be a martingale with respect to histories H_t.

A Discussion of Strong Solutions

- In the case of SDEs, the "unknown" under consideration is a stochastic process. By solving an SDE, we mean determining a process S_t such that the integral equation

$$S_t = S_0 + \int_0^t a(S_u, u)du + \int_0^t \sigma(S_u, u)dW_u, t \in [0, \infty),$$

  is valid for all t. In other words, the evolution of S_t, starting from an initial point S_0, is determined by the two integrals on the right-hand side. The solution process S_t must be such that when these integrals are added together, they should yield the increment S_t - S_0.
- This approach verifies the solution using the corresponding integral equation rather than using the SDE directly, because we do not have a theory of differentiation in stochastic environments.

A Discussion of Strong Solutions (continued)

- Consider a simple ordinary differential equation

$$\frac{dX_t}{dt} = aX_t, \tag{7}$$

  where a is a constant and X_0 is given.
- Suppose the candidate solution

$$X_t = X_0 e^{at} \tag{8}$$

- Then, the solution must satisfy two conditions: First, if we take the derivative of X_t with respect to t,

$$\frac{d}{dt}(X_0 e^{at}) = a[X_0 e^{at}] \tag{9}$$

  Second, at $t = 0$, the function should give a value X_0.

$$(X_0 e^{a0}) = X_0 \tag{10}$$

Verification of Solutions to SDEs

- In SDE, one needs to consider a candidate solution, and then use Ito's Lemma, to see if this candidate satisfies the SDE or the corresponding integral equation. Lets consider the special SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t, t \in [0, \infty), \tag{11}$$

which was used by Black-Scholes (1973) in pricing call options.

- Divide both sides by S_t, we get

$$\frac{1}{S_t} dS_t = \mu dt + \sigma dW_t, t \in [0, \infty), \tag{12}$$

First, we calculate the implied integral equation:

$$\int_0^t \frac{1}{S_u} dS_u = \int_0^t \mu du + \int_0^t \sigma dW_u \tag{13}$$

Verification of Solutions to SDEs (continued)

- The first integral on the right-hand side does not contain any random terms, and it can be calculated in the standard way

$$\int_0^t \mu dt = \mu t \tag{14}$$

- The second integral does contain a random term, but the coefficient of dW_u is a time-invariant constant. Hence, this integral can also be taken in the usual way

$$\int_0^t \sigma dW_u = \sigma[W_t - W_0], \tag{15}$$

where by definition $W_0 = 0$. Thus, we have

$$\int_0^t \frac{1}{S_u} dS_u = \mu t + \sigma W_t. \tag{16}$$

Verification of Solutions to SDEs (continued)

- Consider the candidate

$$S_t = S_0 e^{(a-\frac{1}{2}\sigma^2)t+\sigma W_t}. \tag{17}$$

  Clearly, we are dealing with a strong solution, since S_t depends on W_t and is I_t-adapted.
- Consider calculating the stochastic differential dS_t using Ito's Lemma:

$$dS_t = [S_0 e^{(a-\frac{1}{2}\sigma^2)t+\sigma W_t}]\left[\left(a-\frac{1}{2}\sigma^2\right)dt+\sigma dW_t+\frac{1}{2}\sigma^2 dt\right], \tag{18}$$

  where the very last term on the right-hand side corresponds to the second-order term in Ito's Lemma.
- Canceling similar terms and replacing by S_t, we obtain

$$dS_t = S_t[adt+\sigma dW_t] \tag{19}$$

Linear Constant Coefficient SDEs

- The simplest case of stochastic differential equation is where the drift and diffusion coefficients are independent of the information received over time:

$$dS_t = \mu dt + \sigma dW_t, t \in [0, \infty), \tag{20}$$

where W_t is a standard Wiener process with variance t. The mean of ΔS_t during a small interval of length h is given by

$$E[\Delta S_t] = \mu h \tag{21}$$

The expected variation in ΔS_t will be

$$Var(\Delta S_t) = \sigma^2 h \tag{22}$$

The observations were determined from the iterations

$$S_k = S_{k-1} + 0.01(0.001) + 0.03(\Delta W_k), k = 1, 2, ..., 1000. \tag{23}$$

Linear Constant Coefficient SDEs - Example

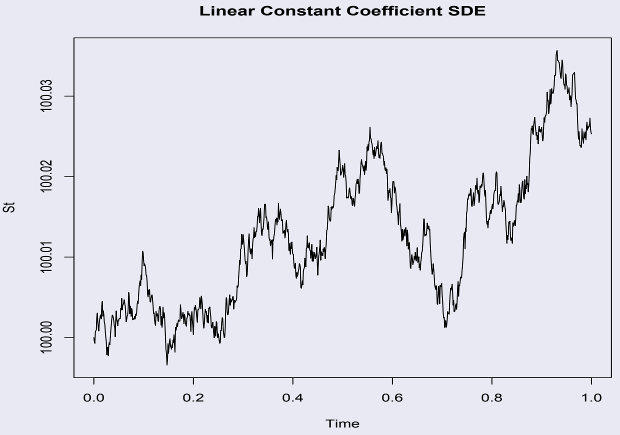

Figure : The behavior of S_t seems to fluctuate around a straight line with slope μ. The size of σ determines the fluctuation around this line.

Geometric SDEs

- The standard SDE used to model underlying asset prices is not the linear constant coefficient model, but is the geometric process. It is the model exploited by Black and Scholes:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, t \in [0, \infty), \tag{24}$$

This model implies that in terms of the formal notation,

$$a(S_t, t) = \mu S_t \tag{25}$$

and

$$\sigma(S_t, t) = \sigma S_t \tag{26}$$

Divide both sides by S_t, we obtain

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{27}$$

$$\text{Example: } dS_t = 0.15 S_t dt + 0.30 S_t dW_t. \tag{28}$$

Geometric SDEs

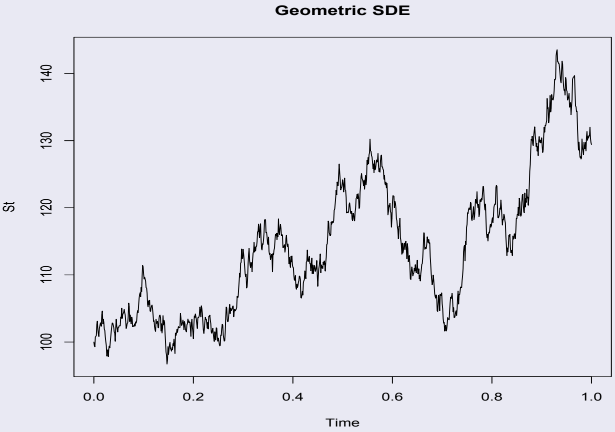

Figure : There is an exponential trend, and random fluctuations around this trend. These variations increase over time because of higher prices.

Square Root SDEs

- A model close the one just discussed is the square root process,

$$dS_t = \mu S_t dt + \sigma \sqrt{S_t} dW_t, t \in [0, \infty), \tag{29}$$

Here the S_t is made to follow an exponential trend, while the standard deviation is made a function of the square root of S_t, rather than of S_t itself. This makes the "variance" of the error term proportional to S_t

We provide an example as following:

$$dS_t = 0.15 S_t dt + 0.30 \sqrt{S_t} dW_t. \tag{30}$$

where the drift and diffusion coefficients are as in the previous case, but where the diffusion is now proportional to $\sqrt{S_t}$ instead of being proportional to S_t.

Square Root SDEs - Example

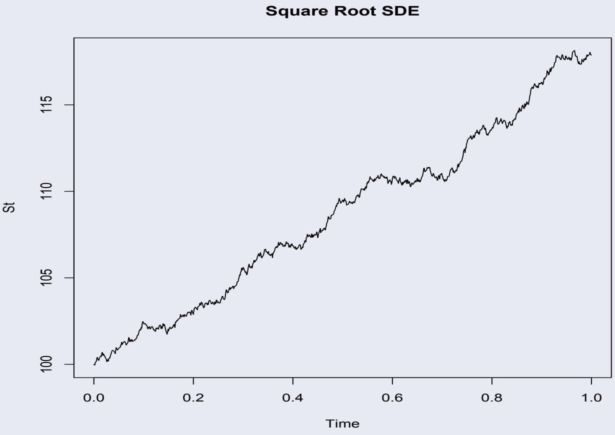

Figure : The fluctuation is more subdued than the ones in the Geometric SDE, yet the sample paths have “similar” trend.

Mean Reverting SDEs

- An SDE that has been found useful in modeling asset prices is the mean reverting model

$$dS_t = \lambda(\mu - S_t)dt + \sigma S_t dW_t, t \in [0, \infty), \tag{31}$$

As S_t falls below some "mean value" μ, the term in parentheses, $(\mu - S_t)$, will become positive. This makes dS_t more likely to be positive. S_t will eventually move toward and revert to the value μ.

A related SDE is the one where the drift is of the mean reverting type, but the diffusion is dependent on the square root of S_t:

$$dS_t = \lambda(\mu - S_t)dt + \sigma\sqrt{S_t}dW_t, t \in [0, \infty), \tag{32}$$

Example:

$$\Delta S_k = .5(.05 - S_{k-1})(.001) + .8\Delta W_k, k = 1, 2, ..., 1000.$$

Mean Reverting - Example

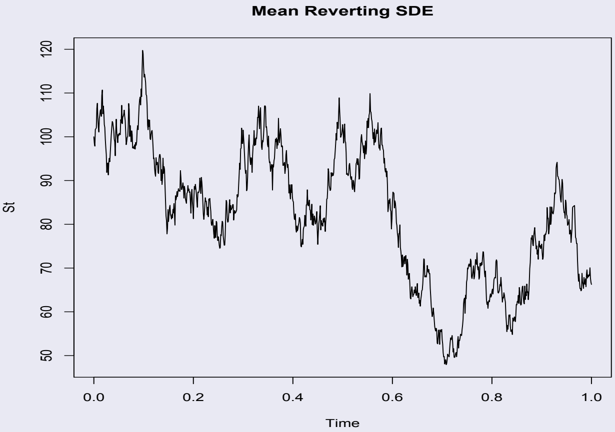

Figure : The mean reverting process has a trend, but the deviation around this trend are not completely random.

Ornstein SDEs

- Another useful SDE is the Ornstein-Uhlenbeck process,

$$dS_t = -\mu S_t dt + \sigma dW_t, t \in [0, \infty), \tag{33}$$

  where $\mu > 0$. Here the drift depends on S_t negatively through the parameter μ, and the diffusion term is of the constant parameter type. Obviously, this is a special case of "mean reverting SDE".
- This model can be used to represent asset prices that fluctuate around zero. The fluctuations can be in the form of excursions, which eventually revert to the long-run mean of zero. The parameter μ controls how long excursions away from this mean will take. The larger the μ, the faster the S_t will go back toward the mean.

$$\Delta dS_k = -.05 S_{k-1} 0.001 + .8 \Delta W_k, k = 1, 2, ..., 1000.$$

Ornstein - Example

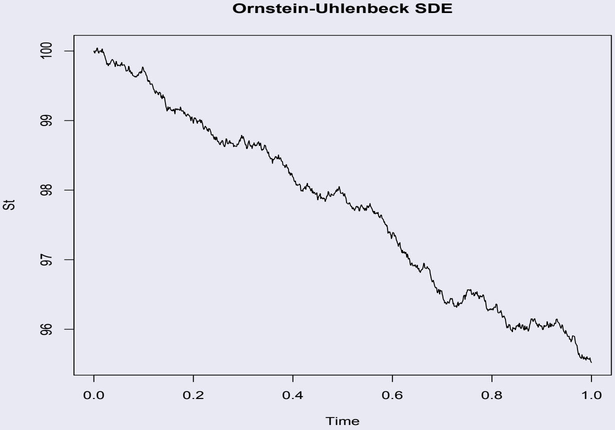

Figure : The fluctuations of the Ornstein process can be in the form of excursions, which eventually revert to the long-run mean of zero.