FE610 Stochastic Calculus for Financial Engineers

Lecture 12. Pricing Derivative Products - Partial Differential Equations

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Outline

Forming Risk-Free Portfolios

- Derivative instruments are contracts written on other securities, and these contracts have finite maturities. At the time of maturity denoted by T, the price F_T of the derivative contract should depend solely on the value of the underlying security S_T, the time T, and nothing else:

$$F_T = F(S_T, T), \tag{1}$$

This implies that at expiration, we know the exact form of the function $F(S_T, T)$. We assume that the same relationship is true for times other than T, and that the price of the derivative product can be written as

$$F(S_t, t). \tag{2}$$

At the outset, a market participant will not know the function form of $F(S_t, t)$ at times other than expiration.

Forming Risk-Free Portfolios (continued)

- If we have an equation describing the way dS_t is determined - then we can use Ito's Lemma to obtain dF_t. This means that dF_t and dS_t would be increments that have the same source of underlying uncertainty. Such relationship makes it possible to form *risk-free portfolios* in countinuous time.
- Let P_t value invested in a combination of $F(S_t, t)$ and S_t:

$$P_t = \theta_1 F(S_t, t) + \theta_2 S_t, \tag{3}$$

  where θ_1 and θ_2 are the quantities of the derivative instrument and the underlying security purchased. They represent portfolio weights.
- The value of this portfolio changes as time t passes because of changes in $F(S_t, t)$ and S_t. Taking θ_1 and θ_2 as constant, we can write:

$$dP_t = \theta_1 dF_t + \theta_2 dS_t. \tag{4}$$

Forming Risk-Free Portfolios (continued)

- Our main interest is in the price of the derivative product, and how this price changes. Thus, we begin by positing a model that determines the dynamics of the underlying asset S_t, and from there we try to determine how $F(S_t, t)$ behaves.
- Accordingly, we assume that the stochastic differential dS_t obeys the SDE

$$dS_t = a(S_t, t)dt + \sigma(S_t, t)dWt, t \in [0, \infty). \tag{5}$$

- Using this, we can apply Ito's Lemma to find dF_t:

$$dF_t = F_t dt + \frac{1}{2} F_{ss} \sigma_t^2 dt + F_s dS_t. \tag{6}$$

- We substitute for dS_t using Eq.(5), and obtain the SDE:

$$dF_t = \left[F_s a_t + \frac{1}{2} F_{ss} \sigma_t^2 + F_t \right] dt + F_s \sigma_t dW_t. \tag{7}$$

Forming Risk-Free Portfolios (continued)

- We first see that the SDE in (7) describing the dynamics of dF_t is driven by the same Wiener increment dW_t that drives the S_t.
- Second, the latter can always be set such that the dP_t is independent of the innovation term dW_t and hence is completely predictable.
- Given that dF_t and dS_t have the same unpredictable component, and given that θ_1 and θ_2 can be set as desired, one can always eliminate the dW_t component. Consider (4) again

$$dP_t = \theta_1 dF_t + \theta_2 dS_t. \tag{8}$$

and substitute dF_t using (6):

$$dF_t = \theta_1 \left[F_s dS_t + \frac{1}{2} F_{ss} \sigma_t^2 dS_t + F_t dt \right] + \theta_2 dS_t. \tag{9}$$

Forming Risk-Free Portfolios (continued)

- In this equation we are free to set θ_1, θ_2 the way we wish. Suppose we ignore for a minute that F_s depends on S_t and select

$$\theta_1 = 1 \text{ and } \theta_2 = -F_s. \tag{10}$$

- These particular values for portfolio weights will lead to cancellation of the terms involving dS_t in (9) and reduces it to

$$dP_t = F_t dt + \frac{1}{2} F_{ss} \sigma_t^2 dt. \tag{11}$$

  Clearly, given the information set I_t, in this expression there is no random term. The dP_t is a completely predictable, deterministic increment for all times t. This means that the portfolio P_t is risk-free.

Forming Risk-Free Portfolios (continued)

- Since there is no risk in P_t, its appreciation must equal the earnings of a risk-free investment during an interval dt in order to avoid arbitrage. Assuming that the (constant) risk-free interest rate is given by r, the expected capital gains:

$$rP_t dt \tag{12}$$

in the case where S_t pays no "dividends", and must equal

$$rP_t dt - \delta dt \tag{13}$$

in the case where S_t pays dividends δ per unit time. In the latter case, the capital gains plus the dividends earned will equal the risk-free rate.

- Utilizing the case with no dividends, we get:

$$rP_t dt = F_t dt + \frac{1}{2} F_{ss} \sigma_t^2 dt. \tag{14}$$

Forming Risk-Free Portfolios (continued)

- Since dt terms are common to all factors, they can be "eliminated" to obtain a partial differential equation:

$$r(F(S_t, t) - F_s S_t) = F_t + \frac{1}{2} F_{ss} \sigma_t^2. \tag{15}$$

We rewrite the last with a simple notation as

$$-rF + rF_s S_t + F_t + \frac{1}{2} F_{ss} \sigma_t^2 = 0, 0 \leq S_t, 0 \leq t \leq T. \tag{16}$$

- We also know at expiration that the price of the derivative product is given by

$$F(S_T, T) = G(S_T, T). \tag{17}$$

where $G(\cdot)$ is a known function of S_T and T. In the case of a call option, we have

$$G(S_T, T) = max[S_T - K, 0]. \text{ K: the strike price} \tag{18}$$

Accuracy of the Method (continued)

- When we develop the SDE using the risk-free portfolio method, we selected the portfolio weights as:

$$\theta_1 = 1, \theta_2 = -F_s. \tag{19}$$

  This selection "works" for constructing a risk-free portfolio, but unfortunately it also violates the assumption that θ_1 and θ_2 are constant. In fact, the θ_2 is now dependent on S_t because, in general, F_s is a function of S_t and t. Thus, first replacing θ_1 and θ_2 with their selected values, and then taking the differential should give a very different results.
- Writing the dependence of F_s on S_t and then differentiating yield:

$$dP_t = (F_t dt + F_s dS_t) - F_s dS_t - S_t dF_s. \tag{20}$$

Accuracy of the Method (continued)

- When we develop the SDE using the risk-free portfolio method, we selected the portfolio weights as:

$$\theta_1 = 1, \theta_2 = -F_s. \tag{21}$$

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- Writing the dependence of F_s on S_t and then differentiating yield:

$$dP_t = (F_t dt + F_s dS_t) - F_s dS_t - S_t dF_s. \tag{22}$$

Accuracy of the Method (continued)

- Note that we now have a third term since the F_s is dependent on S_t and, hence, is time dependent and stochastic. In general, this term will not vanish. In fact, we can use Ito's Lemma and calculate the dF_s, which is a function of S_t and t. This is equivalent to taking the stochastic differential of the derivative's DELTA:

$$dF_s(S_t, t) = F_{st} dt + F_{ss} dS_t + \frac{1}{2} F_{sss} \sigma^2 S_t^2 dt, \qquad (23)$$

where the third derivative of F is there because we are applying Ito's Lemma to the F already differentiated with respect to S_t. After replacing the differential dS_t, and arranging:

$$\begin{aligned} dF_s(S_t, t) &= F_{st} dt + F_{ss}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2} F_{sss} \sigma^2 S_t^2 dt \\ &= \left[F_{st} + F_{ss} \mu S_t + \frac{1}{2} F_{sss} \sigma^2 S_t^2 \right] dt + F_{ss} \sigma S_t dW_t. \qquad (24) \end{aligned}$$

Accuracy of the Method (continued)

- Thus, the formal differential of

$$dP_t = \theta_1 dF(S_t, t) + \theta_2 dS_t, \tag{25}$$

when θ_2 is equal to $-F_s$ will be given by:

$$dP_t = (F_t dt + F_s dS_t) - F_s dS_t$$
$$-S_t \left[\left[F_{st} + F_{ss}\mu S_t + \frac{1}{2} F_{sss}\sigma^2 S_t^2 \right] dt + F_{ss}\sigma S_t dW_t \right]. \tag{26}$$

- This portfolio is self-financing, since we do not have:

$$dP_t = dF(S_t, t) - F_s dS_t. \tag{27}$$

On the right-hand side there are extra terms, and these extra terms will not equal zero unless we have:

$$S_t^2 F_{ss}(\sigma dW_t + (\mu - r)dt) = 0, \tag{28}$$

Accuracy of the Method (continued)

- In order to see this, note that differentiating the Black-Scholes PDE in (16) with respect to S_t again, we can write

$$F_{st} + F_{ss} r S_t + \frac{1}{2} F_{sss} \sigma^2 S_t^2 + \sigma^2 F_{ss} S_t = 0. \tag{29}$$

  Using this equation eliminates most of the unwated terms in (26). But we are still left with:

$$dP_t = (F_t dt + F_s dS_t) - F_s dS_t - S_t dS_t - S_t [F_{ss}(\mu - r) S_t dt] + F_{ss} \sigma S_t^2 dW_t. \tag{30}$$

- Thus, in order to make the portfolio P_t self-financing we need

$$S_t^2 F_{ss}(\sigma dW_t + (\mu - r) dt) = 0, \tag{31}$$

  which will not hold in general.

Accuracy of the Method (continued)

- Although, formally speaking, the risk-free portfolio method is not satisfactory and, in general, makes one work with portfolios that require infusions of cash or leave some capital gains, the method still gives us the correct PDE.
- **How to interpret this result?** The answer is in the additional term, $S_t^2 F_{ss}(\sigma dW_t + (\mu - r)dt)$. This term has nonzero expectation under the true probability P. But once we switch to a risk-free measure $\tilde{P}$ and define a new Wiener process W_t* under this probability, we can write:

$$dW_t* = (\sigma dW_t + (\mu - r)dt). \tag{32}$$

We will have (under synthetic risk measure):

$$E^{\tilde{P}}[S_t^2 F_{ss}(\sigma dW_t + (\mu - r)dt)] \cong 0. \tag{33}$$

It is as if, on the average, self-financing.

Partial Differential Equations

- We rewrite the partial differential equation in a general form, using the shorthand notation $F(S_t, t) = F$,

$$a_0 F + a_1 F_s S_t + a_2 F_t + a_3 F_{ss} = 0, 0 \leq S_t, 0 \leq t \leq T, \tag{34}$$

  with boundary condition

$$F(S_T, T) = G(S_T, T), \tag{35}$$

  $G(\cdot)$ being a known function.
- Forming such risk-free portfolio to obtain arbitrage-free prices for derivatives will always lead to PDEs. The formation of such arbitrage-free portfolios is in general quite straightforward, but the boundary conditions may get more complicated depending on the derivative product we are working with.
- Overall, the method will center on the solution to a PDE.

Questions about PDEs

- **Why is the PDE an "Equation"?**

  Unlike the usual cases in algebra where equations are solved with respect to some variable or vector x, the unknown in our PDE is in the form of a function.

  It is not known what type of function $F(S_t, t)$ represents neither. What is known is that if one takes various partial derivatives of $F(S_t, t)$ and combines them by multiplying by coefficients a_i the result will equal zero.

  Also, at time $t = T$, this function must equal the (known) $G(S_T, T)$ – i.e., it must satisfy the boundary condition.

- **What is the Boundary Condition?**

  In finance, boundary conditions play an important role in determining solutions of a PDE. They represent some contractual clauses of various derivative products.

  The most obvious ones are initial or terminal values.

Classification of PDEs

There are several different ways of classifying PDEs.

- First of all, PDEs can be *linear nonlinear*. This refers to the coefficients applied to partial derivatives in the equation. If an equation is a linear combination of F and its partial derivatives, it is called a linear PDE.
- The second type of classification has to do with the order of differentiation. If all partial derivatives in the equation are first-order, then the PDE will also be first-order. If there are cross-partials, or second partials, then the PDE becomes second-order. For nonlinear financial derivatives such as options, or instruments containing options, the resulting PDE will always be second-order.
- The third type of classification is specific to PDEs. They can be classified as *elliptic, parabolic, or hyperbolic*. The PDEs we encounter in finance are similar to parabolic PDEs.

Example 1: Linear, First-Order PDE

- Consider the PDE for a function $F(S_t, t)$:

$$F_t + F_s = 0, 0 \leq S_t, 0 \leq t \leq T. \tag{36}$$

According to this PDE, the negative of the partial of $F(\cdot)$ with respect to t is equal to its partial with respect to S_t.

- In a financial market, there is no compelling reason why such a relationship should exist between the two partial derivatives. But suppose the equation holds, and we are to find a solution. We can immediately guess a solution:

$$F(S_t, t) = \alpha S_t - \alpha t + \beta, \tag{37}$$

where α, β are any constants. With such a function, the partials will be given by

$$\frac{\partial F}{\partial t} = -\alpha \text{ and } \frac{\partial F}{\partial S_t} = \alpha. \tag{38}$$

Example 1: Linear, First-Order PDE

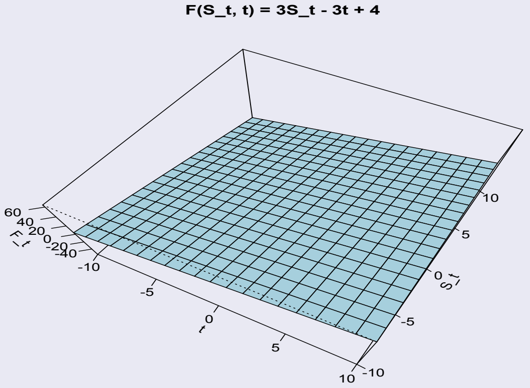

Figure: This figure shows a plane:
$F(S_t, t) = 3S_t - 3t + 4, -10 \leq t \leq 10, -10 \leq S_t \leq 10.$

Example 1: Linear, First-Order PDE

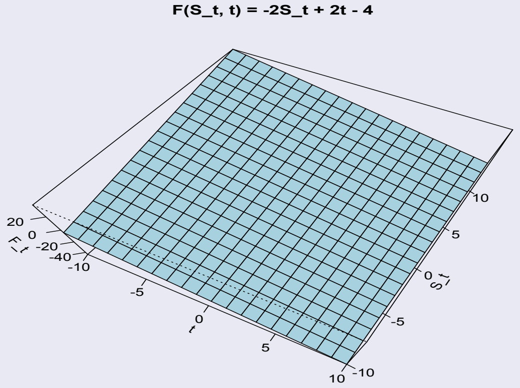

Figure: This figure shows a plane:
$F(S_t, t) = -2S_t + 2t - 4, -10 \leq t \leq 10, -10 \leq S_t \leq 10.$

Example 1: Linear, First-Order PDE

- If in addition to the equation we are given some boundary conditions as well, then we can determine $F(S_t, t)$ precisely. For example, suppose we know that at expiration time $t = 5$ (the boundary for t) we have

$$F(S_5, 5) = 6 - 2S_5. \tag{39}$$

We can now determine the unknown α and β:

$$\alpha = 2 \text{ and } \beta = 4. \tag{40}$$

- [Figure (2)]If we had a second boundary condition, say, at $S_t = 100$, $F(100, t) = 5 + 0.3t$ then there will be no meaningful solution because the two boundary conditions overdetermine the parameters.
- Thus, when $F(S_t, t)$ is a plane, we need a single boundary condition to exactly solve the PDE.

Example 2: Linear, Second-Order PDE

- Now consider a second-order PDE

$$\frac{\partial^2 F}{\partial t^2} = 0.3 \frac{\partial^2 F}{\partial S_t^2},$$
$$\text{or} \quad -0.3F_{ss} + F_{tt} = 0. \tag{41}$$

- First note that we again are dealing with a linear PDE, since the partials in question are combined by using constant coefficients. We may consider a solution as such:

$$F(S_t, t) = \frac{1}{2}\alpha(S_t - S_0)^2$$
$$+\frac{0.3}{2}\alpha(t - t_0)^2 + \beta(S_t - S_0)(t - t_0), \tag{42}$$

  where S_0, t_0 are unknown constants and where the parameters α and β are again unknown.

- If we take the second partials, we can confirm the solution.

Example 2: Linear, Second-Order PDE

- Again, the solution of (42) is not unique, we need boundary conditions. One boundary condition could be at $S_t = 10$:

$$F(10, t) = 100 + t^2. \tag{43}$$

  This is a function that traces a parabola in the F, t plane. But it is no sufficient to determine all the parameters.
- One would need a second boundary condition:

$$F(S_0, 0) = 50 + S_0^2. \tag{44}$$

  This equation is another parabola. But the relevant plane is F, S_t.
- We have a new solution

$$F(S_t, t) = -10(S_t - 4)^2 - 3(t - 2)^2, \quad -10 \leq t \leq 10, -10 \leq S_t \leq 10. \tag{45}$$

Example 1: Linear, First-Order PDE

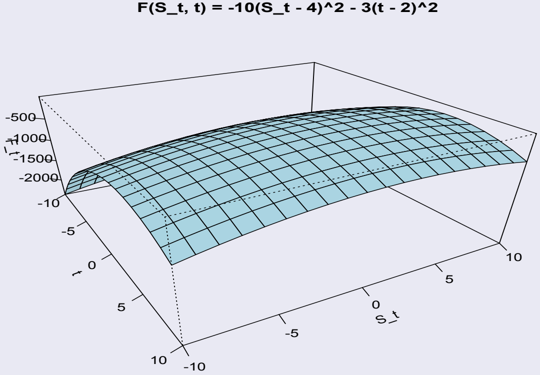

Figure: Suppose we have some boundary conditions:
$F(S_1 0, 10) = -10(S_{10} - 4)^2 - 192$ and $F(0, t) = 160 - 3(t - 2)^2$

A Reminder: Bivariate, Second-Degree Equations

- Let x, y denote two deterministic variables, We can define an equation of the second degree as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{46}$$

Here A, B, C, D, E, and F represent various constants.

(a) **Circle:**

$$A = C \text{ and } B = 0. \tag{47}$$

(b) **Ellipse:**

$$B^2 - 4AC < 0. \tag{48}$$

(c) **Parabola:**

$$B^2 - 4AC = 0. \tag{49}$$

(d) **Hyperbola:**

$$B^2 - 4AC > 0. \tag{50}$$

Types of PDEs

- Now, we can look at partial differential equations of the form

$$a_0 + a_1 F_t + a_2 F_s + a_3 F_{ss} + a_4 F_{tt} + a_5 F_{st} = 0 \qquad (51)$$

are called elliptic PDEs if we have

$$a_5 - 4a_3 a_4 < 0 \qquad (52)$$

, and parabolic, if we have

$$a_5 - 4a_3 a_4 = 0 \qquad (53)$$

, and hyperbolic, if we have

$$a_5 - 4a_3 a_4 > 0 \qquad (54)$$

- Consider a parabolic PDE:

$$F(S_t, t) = -10(S_t - 4)^2 - 3(t - 2). \qquad (55)$$

Types of PDEs

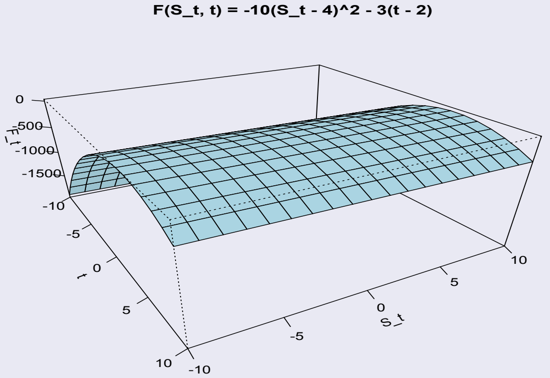

Figure: Such a $F(S_t, t)$ is one of the solutions of PDE: $-\frac{1}{4}F_{ss} + \frac{5}{3}F_t = 0$. The coefficients of the PDE are such that $a_5^2 - 4a_3a_4$ ($a_4 = 0$ and $a_5 = 0$).