

FE610 Stochastic Calculus for Financial Engineers

Lecture 13. The Black-Scholes PDE

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Outline

- 1 The Black-Scholes PDE
- 2 PDEs in Asset Pricing
- 3 Exotic Options
- 4 Solving PDEs in Practice

The Black-Scholes PDE

The Black-Scholes PDE

- Suppose we consider the special SDE where

$$a(S_t, t) = \mu S_t, \text{ and } \sigma(S_t, t) = \sigma S_t, t \in [0, \infty). \quad (1)$$

Under these conditions the fundamental PDE of Black and Scholes and the associated boundary condition are given by

$$-fF + rF_s S_t + F_t + \frac{1}{2} F_{ss} \sigma^2 S_t^2 = 0, 0 \leq S_t, 0 \leq t \leq T \quad (2)$$

$$F(T) = \max[S_T - K, 0]. \quad (3)$$

These two equations are called equations “the fundamental PDE of Black and Scholes (1973).”

The Black-Scholes PDE

- Black and Scholes solve this PDE and obtain the form of the function $F(S_t, t)$ explicitly:

$$F(S_t, t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2), \quad (4)$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad (5)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \quad (6)$$

$N(d_i)$, $i = 1, 2$ are integrals of the normal density:

$$N(d_i) = \int_{-\infty}^{d_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \quad (7)$$

We take the first and second partials of 4 with respect to S_t , and plug these in 3. The result should equal to zero.

PDEs in Asset Pricing

- The partial differential equation obtained by Black and Scholes is relevant under some specific assumptions. There assumptions are:
 - (1) the underlying asset is a stock,
 - (2) the stock does not pay any dividends,
 - (3) the derivative asset is a European style call option that cannot be exercised before the expiration date,
 - (4) the risk-free rate is constant,
 - (5) there are no indivisibilities or transaction costs such as commissions and bid-ask spreads.
- In most applications of pricing, one or more of these assumptions will be violated. If so, in general, the Black-Scholes PDE will not apply and a new PDE should be found. One exception is the violation of assumption (3). If the option is American style, the PDE will remain the same.

Constant Dividends

- Suppose we change one of the Black-Scholes assumptions and introduce a *constant* rate of dividends, δ paid by the underlying asset S_t .

We can try to form the same “approximately” risk-free portfolio by combining the underlying asset and the call option written on it:

$$P_t = \theta_1 F(S_t, t) + \theta_2 S_t. \quad (8)$$

The portfolio weights θ_1, θ_2 can be selected as

$$\theta_1 = 1 \text{ and } \theta_2 = -F_s, \quad (9)$$

so that the “unpredictable” random component is eliminated and a hedge is formed:

$$dP_t = F_t dt + \frac{1}{2} F_{ss} \sigma_t^2 dt. \quad (10)$$

Constant Dividends

The difference occurs in deciding how much this portfolio should appreciate in value. Before, the capital gains were exactly equal to earnings of a risk-free investment. But now, the underlying stock pays a dividend that is predictable at a rate of δ . Hence, the capital gains plus the dividends received must equal the earnings of a risk-free portfolio:

$$dP_t + \delta dt = rP_t dt, \text{ or} \quad (11)$$

$$dP_t = -\delta dt + rP_t dt, \quad (12)$$

Putting this together with (10) we get a slightly different PDE:

$$rF - rF_s S_t - \delta - F_t - \frac{1}{2} F_{ss} \sigma_t^2 = 0. \quad (13)$$

There is a now a constant term δ . Hence stocks paying dividends at a constant rate δ do not present a major problem.

Exotic Options

- Suppose the derivative asset is an option with a possibly random expiration date. For example, there are some “down-and-out” and “up-and-out” options that are known as barrier derivatives.

Unlike “standard options”, the payoff of these instruments also depends on whether or not the spot price of the underlying asset crossed a certain barrier during the life of the option. If such a crossing has occurred, the payoff of the option changes.

- **Lookback Options**

In the standard Black-Scholes case, the call option payoff is equal to $S_T - K$, if the option expires in the money.

In this payoff S_T is the price of the underlying asset at expiration and K is the constant strike price.

Exotic Options (continued)

In the case of a floating lookback call option, the payoff is the difference $S_T - S_{min}$, where S_{min} is the minimum price of the underlying asset observed during the life of the option.

A fixed lookback call option, on the other hand, pays the difference between a fixed strike price K and S_{max} , where the latter is the maximum reached by the underlying asset price during the life of the option.

These options have the characteristic that some positive payoff is guaranteed if the option is in the money during some time over its life.

- Ladder Options

A ladder option has several thresholds, such that if the underlying price reaches these thresholds, the return of the option is “locked in”.

Exotic Options (continued)

- **Trigger or Knock-in Options**

A down-and-in option gives its holder a European option if the spot price falls below a barrier during the life of the option. If the barrier is not reached, the option expires with some rebate as a payoff.

- **Knock-out Options**

Knock-out options are European options that expire immediately if, for example, the underlying asset price falls below a barrier during the life of the option. The option pays a rebate if the barrier is reached. Otherwise, it is a “standard” European option.

- **Other Exotics**

Exotic Options (continued)

- *Basket options*, which are derivatives where the underlying asset is a basket of various financial instruments. Such baskets dampen the volatility of the individual securities. Basket options become more affordable in the case of *emerging market* derivatives.
- *Multi-asset* options have payoffs depending on the underlying price of more than one asset. For example, the payoff of such a call may be

$$F(S_{1T}, S_{2T}, T) = \max[0, \max(S_{1T}, S_{2T}) - K]. \quad (14)$$

Another possibility is the spread call

$$F(S_{1T}, S_{2T}, T) = \max[0, (S_{1T} - S_{2T}) - K]. \quad (15)$$

or the portfolio call (where θ_1 ,

$$F(S_{1T}, S_{2T}, T) = \max[0, (\theta_1 S_{1T} - \theta_2 S_{2T}) - K]. \quad (16)$$

Exotic Options (continued)

- Dual strike call option

$$F(S_{1T}, S_{2T}, T) = \max[0, (S_{1T} - K_1), (S_{2T} - K_2)]. \quad (17)$$

- *Average or Asian options* are quite common and have payoffs depending on the average price of the underlying asset over the lifetime of the option.
- Three major differences between exotics and the standard Black-Scholes case:
 - (1) The expiration value of the option may depend on some event happening over the life of the option. These make the boundary conditions much more complicated than the Black-Scholes case.
 - (2) Derivative instruments may have random expiration dates.
 - (3) The derivative may be written on more than one assets.

All these may lead to changes in the basic PDE that we derived in the Black-Scholes case.

Closed-Form Solutions

- The first method is similar to the one used by Black-Scholes, which involves solving the PDE for a closed-form formula. It turns out that the PDEs describing the behavior of derivative prices cannot in every case be solved for closed forms. In general, either such PDEs are not easy to solve, or they do not have solutions that one can express as closed-form formula.
- The function $F(S_t, t)$ solves a PDE if the appropriate partial derivatives satisfy an equality such as

$$-rF + F_t + rF_s S_t + \frac{1}{2} F_{ss} \sigma^2 S_t^2 = 0, 0 \leq S_t, 0 \leq t \leq T. \quad (18)$$

Now, it is possible that one can find a continuous surface such that the partial derivatives do indeed satisfy the PDE. But, it may be impossible to represent this surface in terms of an easy and convenient formula.

Numerical Solutions

- When a closed-form solution does not exist, a market participant is forced to obtain numerical solutions to PDEs. A numerical solution is like calculating the surface represented by $F(S_t, t)$ directly.

To solve this PDE numerically, one assumes that the PDE is valid for finite increments in S_t and t .

- (1) A grid size for ΔS must be selected as a minimum increment in the price of the underlying security.
- (2) Time t is the second variable in $F(S_t, t)$. Hence, a grid size for Δt is needed as well.
- (3) Next one has to decide on the range of possible values for S_t . Some extreme values should be selected so that observed prices remain within the range $S_{min} \leq S_t \leq S_{max}$.
- (4) Boundary conditions must be determined.
- (5) Assuming that for small but noninfinitesimal ΔS_t and Δt the same PDE is valid, the value of $F(S_t, t)$ at the grid points should be determined.

Numerical Solutions (continued)

- To illustrate, let

$$F_{ij} = F(S_i, j), \quad (19)$$

where F_{ij} is the value at time t_j if the price of the underlying asset is at S_i . The limits of i, j will be determined by the choice of $\Delta S, \Delta t$ and of S_{min}, S_{max} .

- We want to approximate $F(S_t, t)$ at a finite number of options F_{ij} (Figure 1). The dots represent the points at which $F(S_t, t)$ will be evaluated. The sizes of the grids ΔS and Δt determine how “close” these dots will be on the surface.
- We let F_{ij} denote the “dot” that represents the i th value for S_t and the j th value for t . These values for S_t and t will be selected from their respective axes and then “plugged in” to $F(S_t, t)$. The result is written as F_{ij} .

Types of PDEs

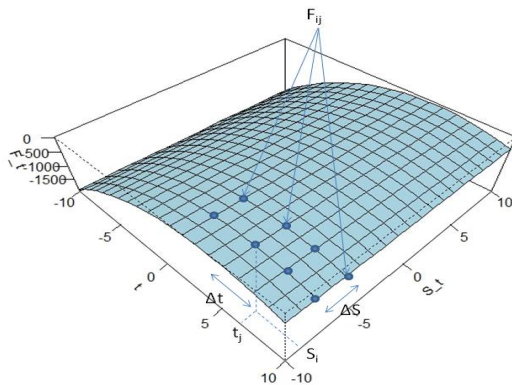


Figure: Solution Surface

Numerical Solutions (continued)

- To carry on this calculation, we need to change the partial differential equation to a difference equation. There are various methods of doing this, each with a different degree of accuracy.

Here, we use the simplest method

$$\frac{\Delta F}{\Delta t} + rS \frac{\Delta F}{\Delta S} + \frac{1}{2} \sigma^2 S^2 \frac{\Delta^2 F}{\Delta S^2} \cong rF, \quad (20)$$

where the first-order partial derivatives are approximated by the corresponding differences.

- For first partials we can use the backward differences

$$\frac{\Delta F}{\Delta t} \cong \frac{F_{ij} - F_{i,j-1}}{\Delta t} \quad (21)$$

and

$$rS \frac{\Delta F}{\Delta S} \cong rS_j \frac{F_{ij} - F_{i,j-1}}{\Delta S} \quad (22)$$

Numerical Solutions (continued)

- We can also use forward differences,

$$\frac{\Delta F}{\Delta t} \cong \frac{F_{i+1,j} - F_{ij}}{\Delta t} \quad (23)$$

$$rS \frac{\Delta F}{\Delta S} \cong rS_j \frac{F_{i+1,j} - F_{ij}}{\Delta S} \quad (24)$$

- For the second-order partials we can use the approximations

$$\frac{\Delta^2 F}{\Delta S^2} \cong \left[\frac{F_{i+1,j} - F_{ij}}{\Delta S} - \frac{F_{ij} - F_{i-1,j}}{\Delta S} \right] \quad (25)$$

where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, N$. The parameters N and n determine the number of points at which we decided to calculate the surface $F(S_t, t)$.

- A system of equations will be solved recursively to obtain values at those points.

Boundary Conditions

- Some of the F_{ij} are known because of endpoint conditions:
- For S_t that is very high, we let $S_t = S_{max}$ and

$$F(S_{max}, t) \cong S_{max} - Ke^{-r(T-t)}. \quad (26)$$

Here, S_{max} is a price chosen so that the call premium is very close to the expiration date payoff.

- For S_t that is very low, we let $S_t = S_{min}$ and

$$F(S_{min}, t) \cong 0. \quad (27)$$

In this case, S_{min} is an extremely low price. There is almost no chance that the option will expire in the money. The resulting call premium is close to zero.

- For $t = T$, we know exactly that

$$F(S_t, T) = \max[S_T - K, 0]. \quad (28)$$