FE610 Stochastic Calculus for Financial Engineers

Lecture 14. Pricing Derivative Products - Equivalent Martingale Measures

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Outline

Translation of Probabilities

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- Recent methods of derivative asset pricing do not necessarily exploit PDEs implied by arbitrage-free portfolios. They rest on converting prices of such assets into martingales.
- This is done through transforming the underlying probability distributions using the tools provided by the Girsanov theorem.

The Girsanov theorem provides the general framework for transforming one probability measure into another "equivalent" measure in more complicated sense. The theorem covers the case of Brownian motion. Hence, the state space is continuous, and the transformations are extended to continuous-time stochastic processes.

The Girsanov Theorem

- The general method can be summarized as follows:
  1. We have an expectation to calculate.
  2. We transform the original probability measure so that expectation becomes easier to calculate.
  3. We calculate the expectation under the new probability.
  4. Once the result is calculated and if desired, we transform this probability back to the original distribution.
- We are given a family of information sets $\{I_t\}$ over a period of $[0, T]$. T is finite. We define a random process ξ_t:

$$\xi_t = e^{(\int_0^t X_u dW_u - \frac{1}{2}\int_0^t X_u^2 du)}, t \in [0, T], \tag{1}$$

where X_t (a Wiener process with probability P) is an I_t-measurable process. We impose an additional condition on X_t (Novikov condition):

$$E[\int_0^t X_u^2 du] < \infty, t \in [0, T]. \tag{2}$$

The Girsanov Theorem (continued)

- It turns out that if the Novikov condition is satisfied, the ξ_t will be a square integrable martingale. Using Ito's Lemma, calculate the differential

$$d\xi_t = [e^{(\int_0^t X_u dW_u - \frac{1}{2}\int_0^t X_u^2 du)}][X_t dW_t], \tag{3}$$

which reduces to

$$d\xi_t = \xi_t [X_t dW_t], \tag{4}$$

Also, we see by simple substitution of $t = 0$ in the random process ξ_t

$$\xi_0 = 1. \tag{5}$$

- Thus, by taking the stochastic integral, we obtain

$$\xi_t = 1 + \int_0^t \xi_s X_s dW_s. \tag{6}$$

The Girsanov Theorem (continued)

But the term $\int_0^t \xi_s X_s dW_s$ is a stochastic integral with respect to a Wiener process. Also, the term $\xi_s X_s$ is I_t-adapted and does not move rapidly. All these imply, as shown before, that the integral is a (square integrable) martingale,

$$E\left[\int_0^t \xi_s X_s dW_s\right] = \int_0^u \xi_s X_s dW_s, u < t \tag{7}$$

- **THEOREM:** If the process ξ_t defined in (1) is a martingale with respect to information sets I_t, and the probability P, then $\tilde{W}_t$ defined by

$$\tilde{W}_t = W_t - \int_0^t X_u du, t \in [0, T], \tag{8}$$

is a Wiener process with respect to I_t and with respect to the probability measure $\tilde{P}_T$, given by

The Girsanov Theorem (continued)

$$\tilde{P}_T = E^P[1_A \xi_T], \tag{9}$$

with A being an event determined by I_T and 1_A being the indicator function of the event.

- In heuristic terms, this theorem states that if we are given a Wiener process W_t, then, multiplying the probability distribution of this process by ξ_t, we can obtain a new Wiener process $\tilde{W}_t$ with probability distribution $\tilde{P}$. The two processes are related to each other through

$$d\tilde{W}_t = dW_t - X_t dt. \tag{10}$$

That is, $\tilde{W}_t$ is obtained by subtracting an I_t-adapted drift from W_t. The main condition for performing such transformations is that ξ_t is a martingale with $E[\xi_T] = 1$.

A Discussion of the Girsanov Theorem

- Suppose the X_u was constant and equaled μ:

$$X_u = \mu. \tag{11}$$

Then, taking the integrals in the exponent in a straightforward fashion, and remembering that $W_0 = 0$,

$$\xi_t = e^{\frac{1}{\sigma^2}[\mu W_t - \frac{1}{2}\mu^2 t]}, \tag{12}$$

which is similar to the $\xi(z_t)$ discussed earlier. This shows the following:

(1) The symbol X_t used in the Girsanov theorem plays the same role μ played in simpler settings. It measures how much the original "mean" will be changed.

(2) In earlier examples, μ was time independent. Here X_t may depend on any random quantity, as long as this random quantity is known by time t (I_t-adapted).

(3) The ξ_t is a martingale with $E[\xi_t] = 1$.

A Discussion of the Girsanov Theorem (continued)

- Consider the Wiener process $\tilde{W}_t$. There is something counter-intuitive about this process. It turns out that both $\tilde{W}_t$ and W_t are standard Wiener processes. Thus, they do not have any drift. But they are related to each other by

$$d\tilde{W}_t = dW_t - X_t dt, \tag{13}$$

which means at least one of them must have nonzero drift.

The point is, $\tilde{W}_t$ has zero drift under $\tilde{P}$, whereas W_t has zero drift under P. Hence, $\tilde{W}_t$ can be used to represent unpredictable errors in dynamic systems given that we switch the probability measures from P to $\tilde{P}$.

- Finally, 1_A is simply a function that has value 1 if A occurs. In fact, we can write:

$$\tilde{P}_T(A) = E^P[1_A \xi_T] = \int_A \xi_T dP \tag{14}$$