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# Distributed Consensus Filter on Directed Graphs with Switching Topologies

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*Abstract*— **We consider distributed estimation on a directed graph with switching topologies. Motivated by a recent PI consensus filter, we modify the protocol and remove the requirement of bi-directional exchange of neighboring gains for fixed topologies. We then extend the protocol to switching topologies and propose a new hybrid consensus filter design. Convergence results under both balanced directed and general directed graphs are given for switching graphs. We finally show satisfactory simulation results.**

*Index Terms*— **Distributed estimation, consensus filter, directed graph, switching topology.**

## I. INTRODUCTION

Distributed estimation is a fundamental problem in networked systems. Direct applications of conventional estimation methods often need all-to-all communications, which causes large communication burdens. Much attention has been paid recently to consensus or gossip algorithms to relax the all-to-all communication requirements to neighbor-to-neighbor communications. In this paper, we present new distributed consensus filter algorithms for directed graphs, and extend it to switching communication topologies.

Average consensus estimations on fixed graphs have been discussed in [1]–[7]. In [1], average consensus protocol is directly applied for the distributed sensor fusion to reach a final estimation with least mean square errors. Although this protocol allows topological switching, it does not have explicit input and cannot track the average of time varying inputs. In more general cases of distributed sensing, each agent has a different input and the goal is to track the average of the set of inputs. In [2]–[4], Olfati-saber and coauthors introduced a distributed low pass consensus filter and a distributed high pass consensus filter, which are able to track the average of inputs to all sensors in a network. In the case that the input to sensors are not identical, estimation error exists even for a set of constant inputs. Progresses were made in [5]–[7] to reduce the estimation error. In [5], [6], Freeman *et. al.* proposed a PI consensus filter, which is able to converge accurately to the average of the inputs when the inputs are time-invariant. Examining the PI filter proposed by [5], [6] in frequency domain, the integral term introduces a zero zero, which cancels out the zero pole introduced by the constant input. This idea is generalized to ideally track the average of time varying inputs by exploiting the internal model principle in [7].

The average consensus protocol converges on switching topologies under some additional mild conditions [8]. In [9], a consensus filter is proposed for distributed map merging under switching topologies. However, the time interval between switching is larger than the convergence time of the filter, *i.e.*, switching happens when the filter almost converges to the desired value. As to arbitrary switching, the system may lose stability even though it is convergent and stable under a fixed topology [10]. In the case of consensus filters with a PI structure, the order of each agent's dynamics increases, which makes it challenging to extend existing convergence results to arbitrary switching.

Motivated by recent advances on consensus filters, we modify the PI consensus filter proposed in [5] and remove the requirement of bi-directional exchange of neighboring gains. We also extend the result to arbitrary switching topologies. By considering switching as a time varying signal under the concept of Dirac delta function and compensating it using a hybrid consensus filter, we rigorously prove convergence under the joint connectivity condition of the switching graph. Simulation results show satisfactory performances.

## II. PRELIMINARIES

### A. Fundamental Knowledge on Graphs

A directed graph $G(V, E, A)$ is denoted by $(V, E, A)$, where $V$ is the set of nodes, $E$ is the set of edges with $E \subseteq V \times V$, and $A = [a_{ij}]$ is the weighted adjacency matrix. The in-degree and out-degree of a node in the directed graph is defined as $\deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji}$ and $\deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}$ respectively. The directed graph $G$ is said to be balanced if the in-degree equals the out-degree for each node in the graph. A special case of balanced graph is undirected graph, which bears the property of $a_{ji} = a_{ij}$ for all $i, j$. A directed graph $G$ is called strongly connected if there always exists a sequence of consecutive edges starting from a given node $i$ to another given node $j$, where node $i$ and node $j$ could be any node in the graph only if $i \neq j$. A directed graph $G$ is called connected if there is an undirected path between any pair of nodes. The degree matrix $\Delta = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ij} = 0$ for all $i \neq j$ and $\Delta_{ii} = \deg_{out}(v_i)$ for

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all $i$. The Laplacian matrix $L$ of the graph $G$ is defined as $L = \Delta - A$. For a directed graph, the rank of its Laplacian matrix is equal to $(n-1)$, where $n$ is the dimension of the Laplacian matrix $L$ if the graph contains a spanning tree. For a balanced graph, containing a spanning tree is equivalent to being connected. A directed graph with topology series $\{G_1(V_1, E_1, A_1), G_2(V_2, E_2, A_2), ...G_k(V_k, E_k, A_k)\}$ with $V_1 = V_2 = ... = V_k = V$, is called jointly-containing-spanning-tree if the union of the topologies $\sum_{i=1}^{k} G_i$, defined as $\sum_{i=1}^{k} G_i = G(V, \bigcup_{i=1}^{k} E_i, \sum_{i=1}^{k} A_k)$, has a spanning tree [11]. Particularly, if all topologies in the topology series are balanced, $\sum_{i=1}^{n} G_i$ having a spanning tree [11] is equivalent to being connected. In this case, this set of topologies is called jointly connected [12].

### B. Graph Centrality

Centrality defines the relative importance of a node on a graph. There are several different measures of centrality, such as degree centrality, betweenness centrality, pagerank centrality [13], [14]. In this paper, we particularly consider the pagerank centrality. The pagerank centrality of node $i$ has the following definition:

$$\alpha(i) d_{out}(v_i) = \sum_{j=1}^{n} w_{ji} \alpha(j) \tag{1}$$

where $\alpha(i) \geq 0$ denotes the pagerank centrality of the $i$th node, $d_{out}(v_i)$ represents the out-degree of node $i$, $w_{ji}$ is the weight for the edge from $i$ to $j$. For all nodes in the graph, the vector $\alpha = [\alpha(1), \alpha(2), ..., \alpha(n)]^T$ satisfies,

$$\alpha^T L = 0 \tag{2}$$

As the Laplacian matrix of a graph always has a zero eigenvalue, Eq. (2) implies the pagerank centrality vector $\alpha$ is the left eigenvector of $L$ corresponding to the zero eigenvalue (referred to as zero left eigenvector from now on). Clearly, the centralities of all nodes on a balanced connected graph are identical since $\mathbf{1} = [1, 1, ..., 1]^T$ is always the zero left eigenvector of $L$ for balanced graphs. In addition, it can be justified that the centrality of a node with zero out-degree equals zero, meaning that this node has no impact to others since there is no out flow from it.

### C. An Existing Consensus Filter

In [5], Freeman et al. proposed an average consensus filter, which reads as follows,

$$\begin{aligned} \dot{x}_i &= -\gamma x_i - \sum_{j \neq i} a_{ij}(x_i - x_j) + \sum_{j \neq i} b_{ji}(\lambda_i - \lambda_j) + \gamma u_i \\ \dot{\lambda}_i &= -\sum_{j \neq i} b_{ij}(x_i - x_j) \end{aligned} \tag{3}$$

where $u_i \in \mathbb{R}$ is the input, $x_i \in \mathbb{R}$ is the decision variable and $\lambda_i \in \mathbb{R}$ is the co-state, $\gamma \in \mathbb{R}^+$ is a constant.

The compact matrix form of this protocol writes:

$$\dot{x} = -L_P x - \gamma(x - u) + L_I^T \lambda \tag{4a}$$

$$\dot{\lambda} = -L_I x \tag{4b}$$

where $u \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$ are the input vector, the decision variable vector and the co-state variable vector, respectively. $L_P$ and $L_I$ are Laplacian matrices constructed by $[a_{ij}]$ and $[b_{ij}]$, respectively.

Note that both $L_I$ and $L_I^T$ appear on the right side of (4) as coefficients (corresponding to the fact that both $b_{ij}$ and $b_{ji}$ appear in (3)). This indicates that "weight information must be communicated between agents in addition to the estimator state values" as claimed in [5]. In this paper, we modify the above consensus filter and remove this requirement of bidirectional communication between agents, so that the algorithm works for directed graphs where bidirectional communication is not possible.

## III. Consensus Filter for Distributed Estimation

In this main section, we present our consensus filter protocols. We first study the case with a fixed topology. We replace $L_I^T \lambda$ in (4) with $\gamma\lambda$ to relax the bi-directional communication requirement and carefully select parameters to guarantee convergence for the modified protocol. Then, the protocol on a directed balanced graph with switching topologies is presented by introducing compensation terms. Finally, the convergence on a general direct graph with switching is analyzed.

### A. Directed Graph with a Fixed Topology

We propose the following protocol by modifying (3),

$$\begin{aligned} \dot{x}_i &= -\sum_{j \in N(i)} w_{ij}(x_i - x_j) - \gamma(x_i - u_i) \\ &\quad -\gamma \int_0^t \sum_{j \in N(i)} w_{ij}(x_i - x_j) dt \end{aligned} \tag{5}$$

where $x_i$ is the decision variable, $u_i$ is the input, $N(i)$ represents the neighbor set of the $i$th node, $w_{ij} \in \mathbb{R}$, $w_{ij} > 0$ if $j \in N(i)$ and otherwise $w_{ij} = 0$, $\gamma \in \mathbb{R}^+$.

Its realization writes:

$$\begin{aligned} \dot{x}_i &= -\sum_{j \in N(i)} w_{ij}(x_i - x_j) - \gamma(x_i - u_i) - \gamma\lambda_i \\ \dot{\lambda}_i &= \sum_{j \in N(i)} w_{ij}(x_i - x_j) \end{aligned} \tag{6}$$

where $\lambda_i$ is the co-state with the initialization $\lambda_i(0) = 0$ for all $i$. The compact form of (5) writes,

$$\dot{x} = -Lx - \gamma(x - u) - \gamma \int_0^t Lx dt \tag{7}$$

where $x$ and $u$ are the decision variable and the input vector, respectively, $L$ is the Laplacian matrix constructed by $[w_{ij}]$.

*Remark 1:* In the protocol realization (6), the update of $x_i$, $\lambda_i$ only requires information from the neighboring set $N(i)$. The communication structure is relaxed to be generally directional by replacing $L_I^T\lambda$ in (4) with $\gamma\lambda$ in (6).

On this protocol, we have the following theorem.

*Theorem 1:* For the consensus protocol (7) or its realization (6) running on a directed graph, $x$ converges to $\frac{\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T u$ for time-invariant $u$, where $\alpha$ is the pagerank centrality of the communication graph (i.e., $\alpha$ is the zero left eigenvector of $L$), provided the graph contains a spanning tree.

*Proof:* Define $\theta = \gamma\int_0^t xdt + x$. The time derivative of $\theta$ is as follows,

$$\begin{aligned}\dot{\theta} &= \gamma x + \dot{x} = \gamma x - Lx - \gamma(x-u) - \gamma\int_0^t Lxdt \\ &= \gamma u - L(x + \gamma\int_0^t xdt) = \gamma u - L\theta \end{aligned} \tag{8}$$

Compared with conventional linear consensus protocol, there is a constant input appears in the dynamics of $\theta$. Calculating the time derivative on both sides of (8) yields, $\ddot{\theta} = -L\dot{\theta}$, which is actually a consensus protocol in terms of the variable $\dot{\theta}$. Therefore, according to the results for linear consensus protocols, we conclude that $\dot{\theta}$ reaches consensus under the condition that the graph has a spanning tree [15], and $\dot{\theta} \to \frac{\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T\dot{\theta}_0$. By multiplying $\alpha^T$ on both sides of (8), we get $\alpha^T\dot{\theta} = \gamma\alpha^T u$. Accordingly, $\dot{\theta} \to \frac{\gamma\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T u$. Defining $\Delta = \dot{\theta} - \frac{\gamma\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T u$, we know $\Delta$ converges to zero. According to the definition of $\theta$, $\dot{x} = -\gamma x + \dot{\theta}$. Defining $z = x - x_e$ where $x_e = \frac{\alpha^T u}{\alpha^T\mathbf{1}}\mathbf{1}$, the dynamics of $z$ can obtained as,

$$\dot{z} = -\gamma z + \Delta \tag{9}$$

We can see that $z$ in (9) is bounded in transition since the linear system (9) is bounded-input-bounded-output (BIBO). Additionally, we can conclude that $z$ in (9) converges to zero as derived as follows based on the final value theorem: $\lim_{t\to\infty} z(t) = \lim_{s\to 0} sF(s)\Delta(s)$, where $F(s) = (sI + \gamma I)^{-1}$ is the transfer function from $\Delta(s)$ to $z(s)$ in complex domain. Note that, $\lim_{s\to 0} s\Delta(s) = \lim_{t\to\infty}\Delta(t) = 0$. All together, we have $\lim_{t\to\infty} z(t) = \lim_{s\to 0} F(s)s\Delta(s) = \lim_{s\to 0} F(s)\lim_{t\to\infty}\Delta(t) = F(0)\times 0 = (\gamma I)^{-1}\times 0 = 0$, which means that $z(t)$ converges to zero. We thus conclude that $x = x_e + z$ is bounded and converges to $x_e = \frac{\alpha^T u}{\alpha^T\mathbf{1}}$. ■

For a balanced graph, whose pagerank centrality is equal for all nodes, Protocol (7) results in an average consensus with equal weights. On this point, the following corollary holds.

*Corollary 1:* For the consensus protocol (7) or its realization (6), $x$ converges to average consensus with the common value $\frac{1}{n}\mathbf{1}^T u$ for time-invariant $u$, provided the graph is balanced and connected.

### *B. Switching Topology for Balanced Graphs*

Switching topology is unavoidable in practical applications. For example, the increase (decrease) of communication power enlarges (narrows down) the neighborhood of a communication device thus changes the communication topology. The design of average consensus filter becomes challenging due to the interaction between the variation of topology and the dynamics of the filter. With the aid of the Dirac delta function, we consider the topological switching as a time varying signal and introduce an extra term to compensate the impact imposed by switching. The resulted consensus filter on switching directed graph is described as a hybrid system.

We first define the following notations following the convention of impulsive and hybrid systems [16]:

- $z(t^+)$ : the right limit of $z(t)$ at $t$, which is defined as $z(t^+) = \lim_{\delta\to 0} z(t+\delta)$ with $\delta > 0$;
- $z(t^-)$ : the left limit of $z(t)$ at $t$, which is defined as $z(t^-) = \lim_{\delta\to 0} z(t-\delta)$ with $\delta > 0$;
- $\Delta z(t)$ : the difference of $z(t)$ at $t$, $\Delta z(t) = z(t^+) - z(t^-)$.

Note that $\Delta z(t)$ equals zero for continuous $z(t)$ while does not equal zero when jump happens for $z(t)$. Following the above notations, we denote the right and the left limits of the Laplacian matrix, and the difference of it, as $L(t^+)$, $L(t^-)$, $\Delta L(t)$, respectively. At the time instant when switching happens, $L(t^+) \neq L(t^-)$ and $\Delta L(t) \neq 0$. The right limit and the left limit of the connection weight $w_{ij}(t)$ at time $t$ is denoted as $w_{ij}(t^+)$ and $w_{ij}(t^-)$, respectively. The difference of $w_{ij}(t^+)$ and $w_{ij}(t^-)$ is denoted as $\Delta w_{ij}(t)$.

With the above notations, we propose our consensus filter on switching directed graphs as follows,

$$\begin{aligned}\dot{x}_i \;=\; & -\sum_{j=1}^{n} w_{ij}(t)(x_i - x_j) - \gamma(x_i - u_i) \\ & + \sum_{k=1}^{m(t)}\sum_{j=1}^{n}\Delta w_{ij}(T(k))\left[x_i(T(k)) - x_j(T(k))\right] \\ & -\gamma\int_0^t\sum_{j=1}^{n} w_{ij}(t)(x_i - x_j)dt \end{aligned} \tag{10}$$

where $x_i = x_i(t)$ represents the $i$th decision variable at time $t$, $T(k)$ denotes the time at which the $k$th topological switching happens, $\Delta w_{ij}(T(k))$ represents the difference of the connection weight of the edge $i-j$ at the switching time $T(k)$. Given time $t$, we can find an integer $m(t) : \mathbb{R}^+ \to \mathbb{I}$, where $\mathbb{I} \subset \mathbb{N}$ is the index set of the switching, such that $T(m(t))$ is the latest time before $t$ when switching happens. Note that $\sum_{j\in N(i)} w_{ij}(x_i - x_j) = \sum_{j=1}^{n} w_{ij}(x_i - x_j)$ since $w_{ij} = 0$ for $j \notin N(i)$.

The realization of this protocol can be written following the notations defined above:

$$\dot{x}_i = -\sum_{j=1}^{n} w_{ij}(t)(x_i - x_j) - \gamma(x_i - u_i) - \gamma\lambda_i + y_i$$
$$\dot{\lambda}_i = \sum_{j=1}^{n} w_{ij}(t)(x_i - x_j)$$
$$\begin{cases} \dot{y}_i = 0 & \text{when no switching} \\ \Delta y_i = \sum_{j=1}^{n} \Delta w_{ij}(t)(x_i - x_j) & \text{switching happens} \end{cases} \quad (11)$$

where $\lambda_i$ and $y_i$ are both co-states for the $i$th node, which are initialized as $\lambda_i(0) = 0$ and $y_i(0) = 0$. $x_i$ is randomly initialized. $\Delta w_{ij}(t) = w_{ij}(t^+) - w_{ij}(t^-)$ and $\Delta y_i = \Delta y_i(t) = y_i(t^+) - y_i(t^-)$ are the difference of the weight $w_{ij}(t)$ and the difference of $y_i(t)$ at the switching time $t$, respectively. The compact form writes:

$$\dot{x} = -L(t)x - \gamma(x-u) - \gamma\int_0^t L(t)xdt + \sum_{k=1}^{m(t)} \Delta L(T(k))x(T(k)) \quad (12)$$

where $L(t)$ represents the Laplacian matrice of the communication graph constructed by weight $[w_{ij}]$ at time $t$, $\Delta L(T(k))$ represents the difference of the Laplacian matrix at time $T(k)$.

*Remark 2:* Comparing the proposed consensus filter (11) for switching topologies with the fixed topology (5), an extra term $y_i$ is added in (11), which is to compensate the effect introduced by the topological switching and is zero all the time for the case without switching (since $y_i$ is initialized to 0).

The following theorem holds for switching balanced graphs.

*Theorem 2:* For the consensus protocol (10) or its realization (11) running on a balanced graph with switching, $x$ converges to $\frac{1}{n}\mathbf{1}\mathbf{1}^T u$ for time-invariant $u$, provided there exists an infinite sequence of uniformly bounded, non-overlapping time intervals, across which the graph is jointly connected.

*Proof:* There are two steps for the proof. The first step uses Dirac delta function to define $\dot{L}(t)$ on switching topologies and simplifies the system using calculus properties. The second step transforms the system using a dynamic transformation and analyzes the system in the new coordinates.

Step 1: To show the idea of compensating the topological switching using $y_i$ in (11), we first use Dirac delta function [17] to define the time derivative of the Laplacian function:

$$\dot{L}(t) = \Delta L(t) \sum_{k=1}^{\infty} \delta(t - T(k)) \quad (13)$$

where $k = 1, 2, ..., \infty$ represents the sequence when the first, second, ..., topological switching happens, $T(k)$ represents the time when the $k$th switching happens, $\delta(t - T(k))$ is the Dirac delta function, whose integration is 1 across the time instant $T(k)$ and equals zero elsewhere. Note that (13) is consistent with the fact that $\dot{L}(t) = 0$ when no switching happens at $t$ and the integration of $\dot{L}(t)$ across $t$ equals $\Delta L(t)$ when switching happens at $t$. According to the so-called sifting property of the Dirac delta function, we can express the summation of the switching signal $\sum_{k=1}^{m(t)} \Delta L(T(k))x(T(k))$ in (12) as follows,

$$\sum_{k=1}^{m(t)} \Delta L(T(k))x(T(k)) = \int_0^t \dot{L}(t)xdt \quad (14)$$

With (14), (12) changes to,

$$\dot{x} = -L(t)x - \gamma(x-u) - \gamma\int_0^t L(t)xdt + \int_0^t \dot{L}xdt \quad (15)$$

According to the rule of integration by parts, we have,

$$\int_0^t \dot{L}(t)xdt = L(t)x - L(0)x(0) - \int_0^t L(t)\dot{x}dt \quad (16)$$

where $L(0)$, $x(0)$ are the Laplacian matrix and the state value at $t = 0$, respectively. With (16), (15) changes to,

$$\dot{x} = -L(t)x - \gamma(x-u) - \gamma\int_0^t L(t)xdt + L(t)x - L(0)x(0) - \int_0^t L(t)\dot{x}dt = -\gamma(x-u) - \gamma\int_0^t L(t)xdt - L(0)x(0) - \int_0^t L(t)\dot{x}dt \quad (17)$$

Step 2: The proof of the second step is similar to the proof of Theorem 1. Similarly, we define new coordinate $\theta = x + \gamma\int_0^t xdt$. Then, we have the dynamics of $\theta$,

$$\dot{\theta} = \dot{x} + \gamma x = \gamma u - \gamma\int_0^t L(t)xdt - L(0)x(0) - \int_0^t L(t)\dot{x}dt = \gamma u - L(0)x(0) - \int_0^t L(t)(\dot{x} + \gamma x)dt = \gamma u - L(0)x(0) - \int_0^t L(t)\dot{\theta}dt \quad (18)$$

Left multiplifying $\mathbf{1}^T$ yields,

$$\mathbf{1}^T\dot{\theta} = \gamma\mathbf{1}^T u \quad (19)$$

On the other hand, recall that $u$ is time-invariant and therefore $\dot{u} = 0$, and calculating time derivative on both sides of (18) yields $\ddot{\theta} = -L(t)\dot{\theta}$, which is exactly a linear consensus protocol with respect to $\dot{\theta}$. According to the results on linear consensus protocol, we know that $\dot{\theta}$ converges to a common value for all elements if the communication topology is jointly connected across an infinite sequence of uniformly bounded, non-overlapping time intervals [18], [19]. With (19), we know this common value equals $\frac{\gamma\mathbf{1}^T u}{n}$, i.e., we conclude that $\lim_{t\to\infty}\dot{\theta} = \frac{\gamma\mathbf{1}^T u}{n}\mathbf{1}$. Let us go back to examine $\theta$ defined as $\dot{\theta} = \dot{x} + \gamma x$. With this equation, $x$ can be regarded as the output of the BIBO system $\dot{x} = -\gamma x + \dot{\theta}$ with $\dot{\theta}$ regarded as the input, which ultimately attenuates to $\frac{\gamma \mathbf{1}^T u}{n}\mathbf{1}$. For such a system, $x$ stabilizes to $\frac{1}{\gamma}\frac{\gamma \mathbf{1}^T u}{n}\mathbf{1} = \frac{\mathbf{1}^T u}{n}\mathbf{1}$, which is the average of input $u$. ■

*Remark 3:* Theorem 2 provides convergence results for a balanced graph with an infinite sequence of jointly connected switching topologies. Actually, for the case that the centrality of each node in the graph is not identical, i.e., the graph is not balanced, but keeps constant over time, i.e., the topologies in the switching family share a common zero left eigenvector $\alpha(t) = \alpha$, average consensus can still be reached, and $x$ converges to $\frac{\alpha^T u}{\alpha^T \mathbf{1}}\mathbf{1}$.

### C. Switching Topology for General Directed Graphs

In the last sub-section, we assumed that the communication graph is balanced for all switching topologies, implying that the centrality of the graph is equal for all nodes all the time, i.e., the zero left eigenvector of the Laplacian matrix $L(t)$ equals $\mathbf{1}$ all the time. This assumption may not be true for a general directed graph with switching topologies. Nevertheless, consensus can still be reached under some mild conditions. On this point, we have the following theorem.

*Theorem 3:* For the consensus protocol (10) or its realization (11) running on a directed graph with switching topologies, $x_i$ converges to consensus for all $i$ for time-invariant $u$, provided that there exists an infinite sequence of uniformly bounded, non-overlapping time intervals, across which the graph is jointly-containing-spanning-tree. The common value of consensus is $\frac{\mu_\infty}{\gamma}$, where

$$\mu_\infty = \frac{\mathbf{1}^T}{n} \lim_{k\to\infty} e^{-L_k(T(k)-T(k-1))} e^{-L_{k-1}(T(k-1)-T(k-2))} \ldots e^{-L_1(T(1)-T(0))}(-L_0 x(0) + \gamma u)$$

and $T(k)$ is the time when the $k$th topology switching happens, $L_k$ is the Laplacian matrix between the time $T(k)$ and $T(k+1)$, $L_0$ is the initial Laplacian matrix, $\gamma > 0$ and $u$ is the constant input.

*Remark 4:* The consensus value in the above theorem, $\frac{\mu_\infty}{\gamma}$, is identical to the steady state common value of the linear consensus protocol $\dot{\mu} = -L(t)\mu$ with the initialization $\mu(0) = -L_0 x(0) + \gamma u$ where $u$ is a constant input.

*Proof:* Define $\theta = x + \gamma \int_0^t x dt$. By following the same procedure as in the proof of theorem 2, we can obtain the dynamics for $\theta$ as $\ddot{\theta} = -L(t)\dot{\theta}$. This is a linear consensus protocol in terms of the variable $\dot{\theta}$. According to results on the linear consensus protocol, we know that $\dot{\theta}$ reaches consensus with time elapse provided that $L(t)$ is jointly-containing-spanning-tree over an infinite sequence of uniformly bounded, non-overlapping time intervals [18], [19]. We denote the consensus value as $\mu_\infty$, i.e., $\lim_{t\to\infty}\dot{\theta} = \mu_\infty \mathbf{1}$. Note that the initial value $\dot{\theta}(0) = \dot{x}(0) + \gamma x(0) = -L(0)x(0) - \gamma(x(0) - u) + \gamma x(0) = -L(0)x(0) + \gamma u$. Solving the linear dynamic system $\ddot{\theta} = -L(t)\dot{\theta}$ relative to $\dot{\theta}$, we obtain that $\lim_{t\to\infty}\dot{\theta} = \lim_{k\to\infty} e^{-L_k(T(k)-T(k-1))} e^{-L_{k-1}(T(k-1)-T(k-2))} \ldots e^{-L_1(T(1)-T(0))} \cdot \dot{\theta}(0)$. Together with the fact that $\lim_{t\to\infty}\dot{\theta} = \mu_\infty \mathbf{1}$, we get the expression of $\mu_\infty$ in the theorem.

Following the definition of $\theta$, we can obtain the dynamics for $x$ as $\dot{x} = -\gamma x + \dot{\theta}$. In this equation, $x$ can be regarded as the output of a BIBO system with $\dot{\theta}$ regarded as the input, which ultimately converges to $\mu_\infty \mathbf{1}$. For such a system, $x$ converges to $\frac{1}{\gamma}\lim_{t\to\infty}\dot{\theta} = \frac{\mu_\infty}{\gamma}\mathbf{1}$. This completes the proof. ■

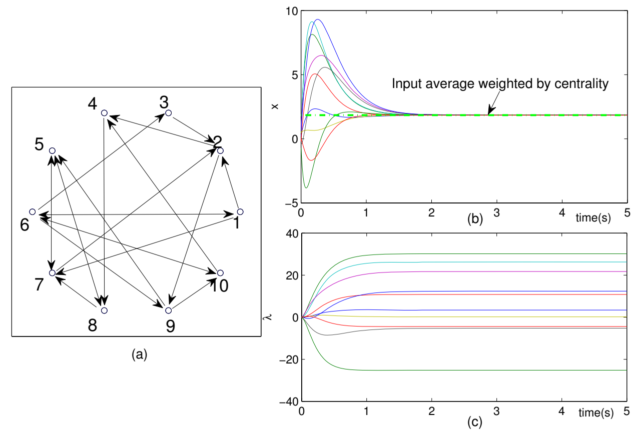


Fig. 1. Protocol (7) on a directed graph with a fixed topology: (a). Graph topology. (b). Time profile of $x(t)$. (c). Time profile of $\lambda(t)$.

## IV. SIMULATIONS

In this section, we use simulations to validate the theoretical conclusions. Both fixed topology and switching topologies will be considered in this section.

We perform simulations on a small scale network with 10 nodes to show the performance (the network topology is shown in Fig. 1). For this graph, the connection weight is set as 1 for existing links. The centrality of this graph can be calculated as $\alpha = [0.6982, 0.0578, 0.0578, 0.0096, 0.0144, 0.6163, 0.0241, 0.0193, 0.1637, 0.3130]$. In the simulation, we choose $\gamma = 4$ and $u = [5.2312, 32.0100, 12.6290, 28.0824, 23.5652, 2.0058, -3.4135, 14.2183, -23.3582, -2.5974]$. As shown in Fig. 1, with a set of random initialization of $x$, $x$ converges to the average of u weighted by $\alpha$ (in this simulation, the weighted average is 1.8405) by running the proposed Protocol (7). The time history of $\lambda$ is also shown in Fig. 1.

For the case with balanced switching topologies, four different topologies switch in the order indicated in Fig. 2 with each topology running for 0.2 seconds. Note that the four topologies are jointly connected but are not connected for a single one. This simulation considers the same input $u$, and the same $\gamma$ as in the fixed topology case. As shown in

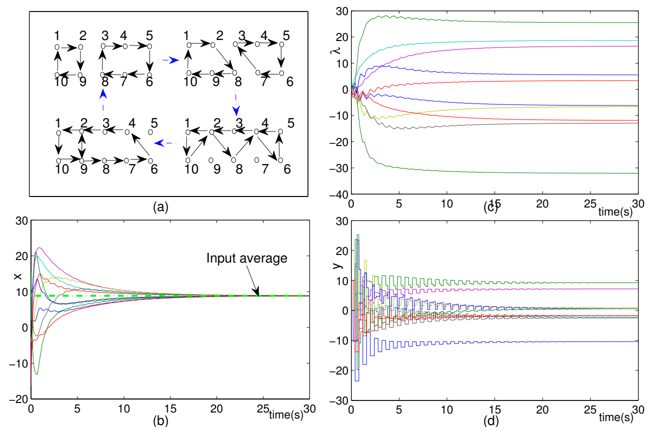


Fig. 2. Protocol (10) on a balanced graph with switching topologies: (a). Graph topologies in the switching family. (b). Time profile of $x(t)$. (c). Time profile of $\lambda(t)$. (d) Time profile of $y(t)$.

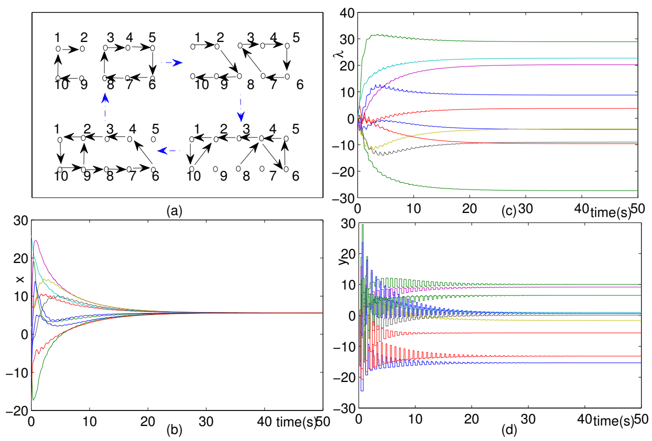


Fig. 3. Protocol (10) on a general direct graph with switching topologies: (a). Graph topologies in the switching family. (b). Time profile of $x(t)$. (c). Time profile of $\lambda(t)$. (d) Time profile of $y(t)$.

Fig. 2, $x$ converges to the desired average by running Protocol (10) under the switching of topologies. The corresponding time evolutions of $\lambda$ and $y$ are also shown in Fig. 2.

By deleting some directed connections on the topologies shown in Fig. 2 (a), a general directed graph with switching topologies is obtained as shown in Fig. 3 (a). Note that none of the four topologies for this graph is balanced. Also note that the graph is jointly-containing-spanning-tree but does not has a spanning tree for any particular topology. With the same input and the same $\gamma$ as in the fixed topology case, and switching the four topologies in the same order as in the balanced switching topology case, $x$ in Protocol (10) converges to consensus, as shown in Fig. 3. The corresponding time evolutions of $\lambda$ and $y$ are also shown in Fig. 3.

## V. Conclusion

In this paper, consensus filters running on a directed graph with switching topologies were investigated. Theoretical results proved the convergence to average consensus of constants inputs under different graph configurations. For directed graphs with a fixed topology, the proposed consensus filter removes the bi-directional communication constraints of an existing PI filter algorithm. Extending to balanced directed graphs with switching topologies, we proposed a hybrid filter to compensate the effect introduced by switching. We finally showed convergence results of the proposed protocol on general directed graphs with switching. Numerical simulations validated theoretical claims.

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