Global Time-Varying Stabilization of Underactuated Surface Vessel

Wenjie Dong and Yi Guo

***Abstract*—This note considers the stabilization problem of an underactuated surface vessel. Three global smooth time-varying control laws are proposed with the aid of different techniques. The first proposed control law makes the state of the closed-loop system asymptotically converge to zero, while the second and the third control laws make the state of the closed-loop system globally exponentially converge to zero. Moreover, the exponential convergence rate of the state of the closed-loop system can be arbitrarily assigned with the third control law. Simulation results show that the proposed control laws are effective.**

***Index Terms*—Global stability, nonlinear control, stabilization, surface vessel, time-varying control, underactuated system.**

I. Introduction

Over the past decade, control of underactuated systems has been one of active research areas in control society. One feature of underactuated systems is that the number of independent actuators of the system is less than that of the degree of freedom. The challenging control problem is how to design a stabilizing control law such that the state of the closed-loop system asymptotically converges to the origin. In the literature, the majority of the research has been focused on control of nonholonomic systems [18] which are special kinds of underactuated systems (i.e., systems with nonintegrable velocity constraints), such as chained systems [18], wheeled mobile robots [7], etc. Nonholonomic systems are drift-less and do not satisfy Brockett's necessary condition for smooth static state feedback control [4]. Therefore, they cannot be stabilized to the origin by any smooth static state feedback law [19], [26]. To overcome this difficulty, novel nonlinear techniques are developed and several interesting control laws are proposed. They are discontinuous state feedback control laws in [1] and [3], time-varying smooth feedback laws in [26] and [27], hybrid feedback laws in [15], etc. For an overview of control of nonholonomic systems, please refer to [14] and the references therein.

Motivated by the challenging theoretic aspect and numerous practical applications, researchers have also attacked the control problem of underactuated systems with nonintegrable dynamics, such as underactuated surface vessel. It is shown in [19], [20] that the underactuated surface vessel does not meet Brockett's necessary condition [4] and therefore cannot be stabilized to the origin by any smooth static state feedback control law. Furthermore, since the model of the underactuated surface vessel is not drift-less, the control methods developed for stabilizing nonholonomic systems cannot be directly used to solve the stabilization problem of underactuated surface vessels. However, with efforts of researchers several stabilizing control laws have been proposed. In [25], a discontinuous feedback control law is proposed based on σ process. With an assumption on the initial state, the state of the closed-loop system exponentially converges to zero with the proposed control law. In [9], discontinuous global stabilizing control laws are proposed with the aid of passivity and Lyapunov theory. In [21] and [23], periodic time-varying feedback control laws are proposed with the aid of averaging techniques. These control laws locally exponentially stabilize the system to the origin. In [6], the controllability of underactuated systems is studied within a geometric framework and a control law is proposed on Lie groups. In [17], a smooth time-varying feedback is proposed for an underactuated surface vessel with the aid of the backstepping technique. It globally stabilizes the system to the origin. In [2], with a special transformation a smooth practical time-varying control law is proposed. This control law makes the state of the system converge to a small ball containing the origin. The tracking control problem of the underactuated surface vessel is also studied recently, several controllers are proposed in [2], [8], [12], and [16].

In this note, we consider the stabilization problem of an underactuated surface vessel. Three stabilizing controllers are proposed with different techniques. First, a new time-varying control law is proposed with the aid of a state transformation and the ideas developed in [27]. This control law globally asymptotically stabilizes the state of the closed-loop system to the origin. Moreover, in order to improve the convergence rate of the state of the closed-loop system, an exponential stabilizing controller is proposed. The proposed controller guarantees the state of the closed-loop system exponentially converges to zero except that the convergence rate cannot be assigned arbitrarily. Finally, to make the convergence rate arbitrarily assigned, a new exponential stabilizing controller is proposed based on a different state transformation and introducing an exponential converging term in one control. With this exponential controller, we can arbitrarily assign the convergence rate of the state of the closed-loop. Three proposed controllers have their own features. They are proposed based on different characters of the system with different techniques. In the first controller, we show how the stabilizing control problem can be solved by combining a suitable state transformation and Barbalat's lemma. In the second and the third controllers, it is shown how the exponential controllers can be obtained by fully utilizing the special structure of the system and the special transformation of some states of the system. By proposing different controllers with different techniques, we can have a better understanding of the nature of the underactuated surface vessel. Compared with the stabilizing control law in [17], our control laws are simple in structure and easily implemented in practice. Compared with the stabilizing control laws in [2], [21], [23], and [25], our control laws are smooth and globally exponentially stabilize the state of the closed-loop system to the origin.

II. Problem Statement

Consider an underactuated surface vessel discussed in [17] and [21]. It has two propellers which are the force in surge and the control torque in yaw. Following the results in [10], the kinematics of the system can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix} \tag{1}$$

where (x, y) denotes the coordinate of the center of mass of the surface vessel in the earth-fixed frame, ψ is the orientation of the vessel, and u, v, and r are the velocities in surge, sway, and yaw, respectively. Assume that: 1) the environment forces due to wind, currents, and waves can be neglected in the model; 2) the inertia, added mass and damping matrices are diagonal, the dynamics of the surface vessel can be written as [10]

$$\begin{cases} \dot{u} = \dfrac{m_{22}}{m_{11}} vr - \dfrac{d_{11}}{m_{11}} u + \dfrac{1}{m_{11}} \tau_1 \\ \dot{v} = -\dfrac{m_{11}}{m_{22}} ur - \dfrac{d_{22}}{m_{22}} v \\ \dot{r} = \dfrac{m_{11} - m_{22}}{m_{33}} uv - \dfrac{d_{33}}{m_{33}} r + \dfrac{1}{m_{33}} \tau_2 \end{cases} \tag{2}$$

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The authors are with the Department of Electrical and Computer Engineering, University of Central Florida, Orlando, FL 32816 USA.

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where $m_{ii}(>0)$ are given by the vessel inertia and the added mass effects, $d_{ii}(>0)$ are given by the hydrodynamic damping, m_{ii} and d_{ii} are assumed to be constants. τ_1 and τ_2 are control inputs.

In this note, we discuss the stabilization problem of system (1)–(2). Obviously, (1)–(2) is underactuated. It is shown in [19] and [20] that there is no smooth static state feedback control law which asymptotically stabilizes the system to the origin. Therefore, stabilizing the state of system (1)–(2) to the origin is challenging. In this note, we propose three new time-varying stabilizing laws with different techniques. The first time-varying control law globally asymptotically stabilizes the system to the origin, while the second and the third time-varying control laws globally exponentially stabilize the system to the origin.

III. Asymptotical Time-Varying Stabilizing Law

To facilitate the control law design, we first transform (1)–(2) into a suitable form. Let the global state transformation [17]

$$\begin{cases} z_1 = x\cos\psi + y\sin\psi \\ z_2 = v \quad z_3 = -x\sin\psi + y\cos\psi + \dfrac{m_{22}}{d_{22}}v \\ z_4 = \psi \quad z_5 = -\frac{m_{11}}{d_{22}}u - z_1 \quad z_6 = r \end{cases} \tag{3}$$

and the control input transformation

$$\begin{cases} w_1 = \left(\dfrac{d_{11}}{d_{22}} - 1\right)u - z_3 z_6 - \frac{\tau_1}{d_{22}} \\ w_2 = \dfrac{m_{11}-m_{22}}{m_{33}}uv - \dfrac{d_{33}}{m_{33}}r + \dfrac{\tau_2}{m_{33}} \end{cases} \tag{4}$$

one has

$$\begin{cases} \dot z_1 = -\dfrac{d_{22}}{m_{11}}z_1 - \dfrac{d_{22}}{m_{11}}z_5 + z_3 z_6 - \dfrac{m_{22}}{d_{22}}z_2 z_6 \\ \dot z_2 = -\dfrac{d_{22}}{m_{22}}z_2 + \dfrac{d_{22}}{m_{22}}z_6(z_1+z_5) \end{cases} \tag{5}$$

$$\begin{cases} \dot z_3 = z_5 z_6 \quad \dot z_4 = z_6 \\ \dot z_5 = w_1 \quad \dot z_6 = w_2. \end{cases} \tag{6}$$

Lemma 1: State transformation (3) is a global diffeomorphism. $\lim_{t\to\infty} z_i = 0 (1 \le i \le 6)$ imply that (x, y, ψ, u, v, r) converges to zero as time tends to infinite.

Proof: It is easy to verify that the state transformation (3) is a global diffeomorphism by definition [13]. Furthermore, $\lim_{t\to\infty} z_i = 0 (1 \le i \le 6)$ imply that (x, y, ψ, u, v, r) converges to zero as time tends to infinite. ∎

Since the state transformation (3) is a global diffeomorphism, system (1)–(2) represents system (5)–(6). Therefore, it is only needed to discuss the stabilization problem of system (5)–(6). For the system (5)–(6), one has the following lemma.

Lemma 2: For (5), if z_3, z_5, and z_6 converge to zero as time tends to infinite, then $\lim_{t\to\infty} z_i(t) = 0 (i = 1, 2)$.

Proof: Assume z_3, z_5, and z_6 converge to zero as time tends to infinite. Let the nonnegative function

$$V = \frac{d_{22}}{2m_{22}}z_1^2 + \frac{m_{22}}{2d_{22}}z_2^2 \tag{7}$$

differentiating V along (5), one has

$$\dot V = -\frac{d_{22}^2}{m_{11}m_{22}}z_1^2 - z_2^2 - \frac{d_{22}^2}{m_{11}m_{22}}z_1 z_5 + \frac{d_{22}}{m_{22}}z_1 z_3 z_6 + z_2 z_5 z_6 \le -c_1 V + c_2(t)\sqrt{V} \tag{8}$$

where

$$c_1 = 2\min\left\{\frac{d_{22}}{m_{22}}, \frac{d_{22}}{m_{11}}\right\}$$
$$c_2(t) = \sqrt{\frac{2d_{22}}{m_{22}}}\left(\left|z_3(t)z_6(t) - \frac{z_5(t)}{m_{11}}\right| + |z_5(t)z_6(t)|\right). \tag{9}$$

Let $\gamma = \sqrt{V}$, (8) can be written as

$$\dot\gamma \le -\frac{c_1}{2}\gamma + \frac{1}{2}c_2(t). \tag{10}$$

Since $c_1 > 0$ and c_2 is bounded and converges to zero, γ is bounded and converges to zero. Thus, V is bounded and converges to zero. Furthermore, z_1 and z_2 are bounded and converge to zero. ∎

Noting the result in Lemma 2, it is only needed to design a stabilizing control law for (6). In the following, a time-varying control law is proposed for (6) in two steps with the aid of the well-known backstepping technique [11]. In the first step, we consider the stabilization problem of the subsystem (z_3, z_4, z_5). Assume z_6 is a virtual control, we design z_6 and w_1 such that z_3, z_4 and z_5 globally asymptotically converge to zero. The result of the first step is stated in the following lemma.

Lemma 3: In (6), assume z_6 is a virtual control, let

$$z_6 = \alpha \quad w_1 = -k_2 z_3\alpha - k_1 z_5 \tag{11}$$

where

$$\alpha = -k_3 z_4 + z_3\cos t \tag{12}$$

constants $k_i > 0 (1 \le i \le 3)$, then z_3, z_4, and z_6 asymptotically converge to zero.

Proof: By the values of z_6 and w_1, the first three equations in (6) are

$$\begin{cases} \dot z_3 = z_5\alpha \\ \dot z_4 = -k_3 z_4 + z_3\cos t \\ \dot z_5 = -k_2 z_3\alpha - k_1 z_5. \end{cases} \tag{13}$$

Let

$$V_1 = \frac{1}{2}\left(k_2 z_3^2 + z_5^2\right)$$

differentiating V_1 along (13) one obtains

$$\dot V_1 = -k_1 z_5^2 \le 0.$$

Since V_1 is nonincreasing, V_1 converges to some limit value $V_{1\lim}(\ge 0)$. Therefore, $\lim_{t\to\infty} z_5 = 0$ by the results in [13]. By the boundedness of V_1, z_3 and z_5 are bounded. Noting the second equation of (13), z_4 is bounded.

Since α is bounded, $\alpha^2 z_5$ converges to zero. Differentiating $\alpha^2 z_5$, one has

$$d(\alpha^2 z_5)/dt = -k_2 z_3\alpha^3 - k_1 z_5\alpha^2 + 2z_5\alpha\dot\alpha$$

where the first term on the right-hand side is uniformly continuous, the other two terms in the right hand converge to zero. By Barbalat's lemma, $z_3\alpha^3$ converges to zero. Furthermore, $z_3\alpha$ converges to zero.

Next, we prove that α converges to zero by contradiction. Assume α does not converge to zero, then $\alpha V_{\lim}$ converges to zero because αz_3 and αz_5 converge to zero. Therefore, $V_{1\lim} = 0$. Thus, z_3 converges to zero. By the second equation in (13), z_4 converges to zero. So, α converges to zero, which contradicts to the assumption that α does not converge to zero. Therefore, α converges to zero. Noting

$$\dot\alpha = -z_3\sin t + (\dot z_3\cos t - k_3\dot z_4)$$

where the first term in the right hand of the equation is uniformly continuous, the other terms converge to zero. By Barbalat's lemma, $z_3\sin t$ converges to zero. Furthermore $z_3\sin^2 t$ converges to zero. Differentiating $z_3\sin t$ one can easily prove that $z_3\cos t$ converges to zero with the aid of Barbalat's lemma. Therefore, $z_3\cos^2 t$ converges to zero. So $z_3 (= z_3\sin^2 t + z_3\cos^2 t)$ converges to zero. Noting the second equation in (13), z_4 converges to zero. ∎

In the second step, we design the actual control w_2 such that z_3, z_4, z_5, and z_6 asymptotically converge to zero with the aid of the result in Lemma 3. One has the following result.

Lemma 4: For system (6), the control inputs

$$w_1 = -k_2 z_3 \alpha - k_1 z_5 \tag{14}$$

$$w_2 = -k_4(z_6 - \alpha) + \dot{\alpha} - k_2 z_3 z_5 \tag{15}$$

globally asymptotically stabilize z_3, z_4, z_5, and z_6 to the origin, where constants $k_i > 0 (1 \le i \le 4)$.

Proof: Let $\tilde{z}_6 = z_6 - \alpha$, the closed-loop system of (6) is written as

$$\begin{cases} \dot{z}_3 = z_5 \alpha + z_5 \tilde{z}_6 \\ \dot{z}_4 = -k_3 z_4 + z_3 \cos t + \tilde{z}_6 \\ \dot{z}_5 = -k_2 z_3 \alpha - k_1 z_5 \\ \dot{\tilde{z}}_6 = -k_2 z_3 z_5 - k_4 \tilde{z}_6. \end{cases} \tag{16}$$

Let the nonnegative function

$$V_2 = \frac{1}{2}\left(k_2 z_3^2 + z_5^2 + \tilde{z}_6^2\right)$$

differentiating it along (16) one obtains $\dot{V}_2 = -k_1 z_5^2 - k_4 \tilde{z}_6^2$. Therefore, V_2 is nonincreasing and converges to a limit value $V_{2\lim} (\ge 0)$. Therefore, z_3, z_5, and $\tilde{z}_6$ are bounded. Noting the second equation in (16), z_4 is bounded. Furthermore, α and z_6 are bounded. Noting z_5^2 and $\tilde{z}_6^2$ are uniformly continuous, by Barbalat's lemma z_5^2 and $\tilde{z}_6^2$ converge to zero. Following the proof in Lemma 3, one can prove that $z_3 \alpha$ and α converge to zero. Furthermore, it can be proved that z_3 and z_4 converge to zero, respectively. Therefore, z_6 converges to zero too. ■

Based on the results in Lemma 2 and Lemma 4, one has the following theorem.

Theorem 1: For (5)–(6), the control law (14)–(15) ensures that $z_i (1 \le i \le 6)$ globally asymptotically converge to zero, where the control parameters $k_i > 0 (1 \le i \le 4)$.

In Theorem 1, a global stabilizing time-varying control law is proposed. In the control law, there are only four control parameters (i.e., k_1, k_2, k_3, and k_4). In order to make the state of the closed-loop system converge to zero, they are only needed to be positive. Generally, large values of $k_i (0 \le i \le 2)$ result in fast convergence of the system state. However, there is a tradeoff in selecting the values of k_1, k_2, and k_3. If k_1 and k_3 are chosen to be large positive constants, z_3 may converge to zero slowly. Also, k_1 and k_3 cannot be too small, otherwise z_5 and z_4 will converge to zero slowly. In practice, the control parameters should be selected according to control objectives. In [17], a time-varying control law is also proposed. However, control law (14)–(15) is simple in structure and easily implemented in practice.

Control law (14)–(15) guarantees that the state of the closed-loop system converges to zero, however the convergence rate is slow. In order to improve the convergence rate of the state of the closed-loop system, two exponentially stabilizing control laws are proposed in the next section.

IV. Time-Varying Exponentially Stabilizing Laws

A. Exponential Law Based on Transformation (3)

With the aid of the transformations (3)–(4), one has the following lemma.

Lemma 5: The state of (5) exponentially converges to zero if the state of (6) exponentially converges to zero.

Proof: Assume that $z_i (3 \le i \le 6)$ exponentially converge to zero. Let V be defined in (7), differentiating it along system (5), one has (8) with c_1 and c_2 defined in (9). Let $\gamma = \sqrt{V}$, one has (10). Noting $c_1 > 0$ and c_2 exponentially converges to zero, so γ exponentially converges to zero. Therefore, V exponentially converges to zero. Furthermore, z_1 and z_2 exponentially converge to zero. ■

By Lemma 5, it is only needed to consider the exponential stabilization problem of (6). For (6), if one chooses

$$w_2 = -(k_1 + k_2) z_6 - k_1 k_2 z_4 - (\lambda - k_1)(\lambda - k_2) e^{-\lambda t} \tag{17}$$

where constants $\lambda > 0, k_1 > \lambda, k_2 > \lambda$, and $k_1 \ne k_2, z_4$, and z_6 are as follows:

$$z_4 = a_1 e^{-k_1 t} + a_2 e^{-k_2 t} - e^{-\lambda t}$$
$$z_6 = -k_1 a_1 e^{-k_1 t} - k_2 a_2 e^{-k_2 t} + \lambda e^{-\lambda t}$$

where

$$a_1 = \frac{k_2 z_4(0) + z_6(0) + k_2 - \lambda}{k_2 - k_1}$$
$$a_2 = \frac{k_1 z_4(0) + z_6(0) + k_1 - \lambda}{k_1 - k_2}.$$

Obviously, z_4 and z_6 globally exponentially converge to zero. Let $\bar{z}_3 = e^{\lambda t} z_3$, the subsystem $(\bar{z}_3, z_5)$ can be written as

$$\dot{q} = (A + A_1) q + b\omega \tag{18}$$

where $q = [\bar{z}_3, z_5]^T, b = [0, 1]^T, \omega = w_1$, and

$$A = \begin{bmatrix} \lambda & \lambda \\ 0 & 0 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0 & -(k_1 a_1 e^{-(k_1 - \lambda)t} + k_2 a_2 e^{-(k_2 - \lambda)t}) \\ 0 & 0 \end{bmatrix}.$$

Lemma 6: [13]: For the system

$$\dot{q} = (\Lambda + \Lambda_1(t)) q \tag{19}$$

where Λ is a constant matrix, and $\Lambda_1(t)$ is a time-varying matrix. If Λ is Hurwitz stable and $\Lambda_1(t)$ is bounded and $\|\Lambda_1(t)\|$ exponentially converges to zero as t approaches to infinity, then q is bounded and exponentially converges to zero.

If one chooses

$$w_1 = -Kq = -k_3 e^{\lambda t} z_3 - k_4 z_5 \tag{20}$$

where constant vector $K = [k_3, k_4]$. If $k_3 > k_4 > \lambda, A - bK$ is Hurwitz stable. Noting A_1 exponentially converges to zero, q exponentially converges to zero by Lemma 6. Noting the relation between z_3 and $\bar{z}_3, z_3$ exponentially converges to zero. Therefore, one has the following result.

Lemma 7: For (6), if the control inputs are chosen as (20) and (17), the state of (6) exponentially converges to the origin, where control parameters $\lambda > 0, k_1 > \lambda, k_2 > \lambda (k_2 \ne k_1)$, and $k_3 > k_4 > \lambda$.

With the aid of Lemmas 3 and 7, one has the following theorem.

Theorem 2: For (5), if the control inputs are chosen as (20) and (17), $z_i (1 \le i \le 6)$ globally exponentially converge to zero, where control parameters $\lambda > 0, k_1 > \lambda, k_2 > \lambda (k_2 \ne k_1)$, and $k_3 > k_4 > \lambda$.

In the control laws (20) and (17), the control parameters are $k_i (1 \le i \le 4)$ and λ. If $\lambda > 0, k_1 > \lambda, k_2 > \lambda, k_1 \ne k_2$, and $k_3 > k_4 > \lambda$, the state of the closed-loop system (5) globally exponentially converges to zero. The exponential convergence rate of the states $z_i (3 \le i \le 6)$ can be adjusted by the control parameters λ and $k_i (1 \le i \le 4)$. The exponential convergence rate of the states z_1 and z_2 depends on d_{22}, m_{11}, m_{22} and the exponential convergence rate of z_3 and z_5, which means the exponential convergence rate of z_1 and

z_2 cannot be arbitrarily assigned because d_{22}, m_{11} and m_{22} cannot be changed by the control law for the given system.

B. Stabilizing Law With Arbitrary Exponential Convergence Rate

With the idea in the last subsection, another exponential control law is proposed based on a new state transformation in this subsection. This new control law can make the state of the closed-loop system exponentially converge to zero with any desired rate. Let the state transformation

$$\begin{cases} z_1 = \psi \quad z_2 = r \quad z_3 = -x\sin\psi + y\cos\psi \\ z_4 = v \quad z_5 = x\cos\psi + y\sin\psi \quad z_6 = u \end{cases} \tag{21}$$

and the control input transformation

$$\begin{cases} w_1 = \frac{m_{11}-m_{22}}{m_{33}}uv - \frac{d_{33}}{m_{33}}r + \frac{\tau_2}{m_{33}} \\ w_2 = \dfrac{m_{22}}{m_{11}}vr - \dfrac{d_{11}}{m_{11}}u + \dfrac{\tau_1}{m_{11}} \end{cases} \tag{22}$$

one has

$$\begin{cases} \dot{z}_1 = z_2 \quad \dot{z}_2 = w_1 \quad \dot{z}_3 = z_4 - z_5 z_2 \\ \dot{z}_4 = -cz_6 z_2 - dz_4 \quad \dot{z}_5 = z_6 + z_3 z_2 \quad \dot{z}_6 = w_2 \end{cases} \tag{23}$$

where $c = m_{11}/m_{22}$ and $d = d_{22}/m_{22}$. Let the control input

$$w_1 = -(k_1 + k_2)z_2 - k_1 k_2 z_1 - (\alpha - k_1)(\alpha - k_2)e^{-\alpha t}/\alpha \tag{24}$$

where $\alpha > 0, k_1 > \alpha, k_2 > \alpha$, and $k_1 \neq k_2$, one has

$$\begin{aligned} z_1 &= (k_2 z_1(0) + z_2(0) - 1 + k_2/\alpha)e^{-k_1 t}/(k_2 - k_1) \\ &\quad + (k_1 z_1(0) + z_2(0) - 1 + k_1/\alpha)e^{-k_2 t}/(k_1 - k_2) - e^{-\alpha t}/\alpha \\ z_2 &= -(k_2 z_1(0) + z_2(0) - 1 + k_2/\alpha)k_1 e^{-k_1 t}/(k_2 - k_1) \\ &\quad - (k_1 z_1(0) + z_2(0) - 1 + k_1/\alpha)k_2 e^{-k_2 t}/(k_1 - k_2) + e^{-\alpha t}. \end{aligned}$$

Therefore, z_1 and z_2 exponentially converge to zero with rate α.

Let $\bar{z}_3 = e^{\alpha t} z_3$ and $\bar{z}_4 = e^{\alpha t} z_4$, then

$$\begin{cases} \dot{\bar{z}}_3 = \alpha \bar{z}_3 + \bar{z}_4 - z_5 + z_5 \delta(t) \\ \dot{\bar{z}}_4 = (\alpha - d)\bar{z}_4 - cz_6 + cz_6\delta(t) \\ \dot{z}_5 = z_6 + \bar{z}_3(e^{-2\alpha t} + e^{-2\alpha t}\delta(t)) \\ \dot{z}_6 = w_2 \end{cases} \tag{25}$$

where

$$\begin{aligned} \delta(t) = &-(k_2 z_1(0) + z_2(0) - 1 + k_2/\alpha)k_1 e^{-(k_1-\alpha)t}/(k_2 - k_1) \\ &-(k_1 z_1(0) + z_2(0) - 1 + k_1/\alpha)k_2 e^{-(k_2-\alpha)t}/(k_1 - k_2). \end{aligned}$$

Or in a compact form (18) with $q = [\bar{z}_3, \bar{z}_4, z_5, z_6]^T, b = [0, 0, 0, 1]^T, \omega = w_2$

$$A = \begin{bmatrix} \alpha & 1 & -1 & 0 \\ 0 & \alpha - d & 0 & -c \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & \delta(t) & 0 \\ 0 & 0 & 0 & c\delta(t) \\ e^{-2\alpha t} + e^{-2\alpha t}\delta(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, if $\alpha \neq d$, (A, b) is controllable. A_1 exponentially converges to zero. Let the control input

$$w_2 = -Kq \tag{26}$$

where $K = [k_3, k_4, k_5, k_6]$ are chosen such that $A - bK$ is Hurwitz stable. By Lemma 6, q exponentially converges to zero. Therefore, $z_i (3 \leq i \leq 6)$ exponentially converge to zero. One has the following theorem.

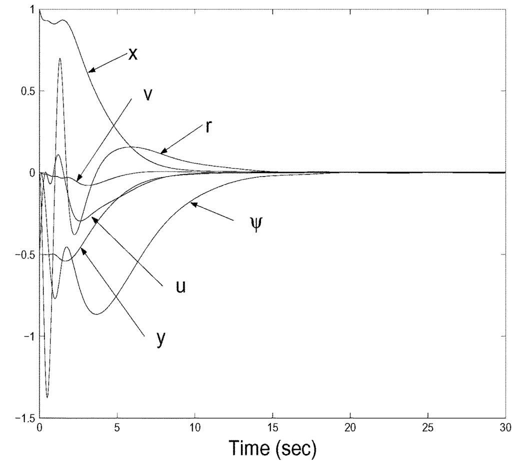

Fig. 1. Response of the state.

Theorem 3: For (23), the control inputs (24) and (26) exponentially stabilize the state of the closed-loop system to zero, where the control parameters $\alpha > 0 (\alpha \neq d), k_1 > \alpha, k_2 > \alpha (k_2 \neq k_1)$, and K is chosen such that $A - bK$ is Hurwitz stable.

If $\alpha \neq d$, the eigenvalues of $A - bK$ can be arbitrarily assigned. If one quires the state of the closed-loop system to exponentially converge to zero with the least exponential rate β, one can select $\alpha > \beta (\alpha \neq d), k_1 > \alpha + \beta, k_2 > \alpha + \beta (k_2 \neq k_1), K$ such that the real parts of the eigenvalues of $A - bK$ is less than $-\beta$.

Several other exponential stabilizing control laws are proposed in the literature. In [25], a discontinuous exponential control law is proposed. In [21] and [23], local exponential laws are proposed but the size of the attraction region is unknown. In [22] and [2], practical exponential control laws are proposed. Compared with the existing exponential control laws, our exponential control laws exponentially stabilize the system to the origin and are global. Especially, the control law (24)–(26) make the state of the closed-loop system exponentially converge to zero with prescribed rate.

V. Simulation

In this section, we study the effectiveness of the proposed control laws by simulation. Consider an underactuated surface vessel with the model parameters [25]: $m_{11} = 200$ kg, $m_{22} = 250$ kg, $m_{33} = 80$ kg, $d_{11} = 70$ kg/s, $d_{22} = 100$ kg/s, and $d_{33} = 50$ kg/s. Assume the initial condition is $(x(0), y(0), \psi(0), u(0), v(0), r(0)) = (1, -0.5, 0, 0, 0, 0)$.

We first use the control law (14)–(15) to asymptotically stabilize the closed-loop system. The control parameters are chosen as $k_1 = 0.5, k_2 = 15, k_3 = 0.5$, and $k_4 = 0.5$. Fig. 1 shows that the states of the closed-loop system converge to zero. Fig. 2 shows that the states of the system do not exponentially converge to zero because the logarithms of the absolute value of each state do not decay linearly. Fig. 3 shows that the two control inputs are bounded and converge to zero. The simulation results verify the result in Theorem 1.

In the second simulation, the exponentially stabilizing law (20) and (17) is applied to (5)–(6) with the model parameters and the initial condition stated above. In the control law, we choose $\lambda = 0.5, k_1 = 1, k_2 = 1.5, k_3 = 3.9$, and $k_4 = 2.3$. Fig. 4 shows that the states of the closed-loop system converge to zero. Furthermore, Fig. 5 shows that all states exponentially converge to zero, which confirms the results in Theorem 2. Fig. 6 shows that the control inputs are bounded and converge to zero.

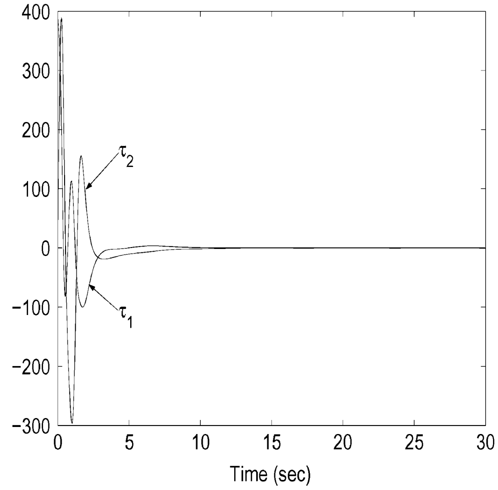

Fig. 2. Control inputs.

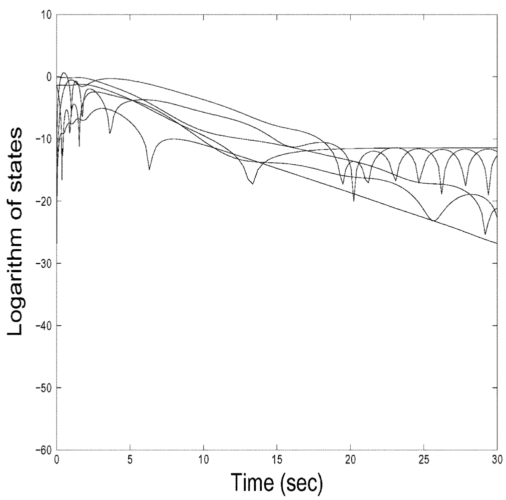

Fig. 3. Logarithm of the absolute values of each state.

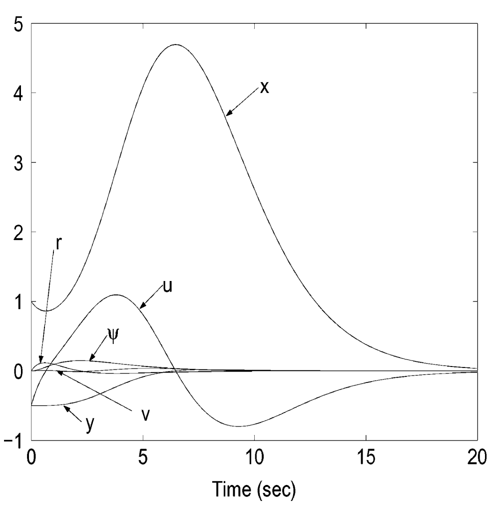

Fig. 4. Response of the state.

In the third simulation, the exponentially stabilizing law (24)–(26) is applied to (5)–(6) with the model parameters and the initial condition stated previously. In the control law, we choose $\alpha = 0.5, k_1 = 1.7, k_2 = 2, k_3 = -12.1, k_4 = 77, k_5 = 74.6$, and $k_6 = 5.7$ (i.e., the eigenvalues of $A - bK$ are $-1.5, -1.2, -1.4$, and -1). Fig. 7 shows that the states of the closed-loop system converge to zero. Furthermore,

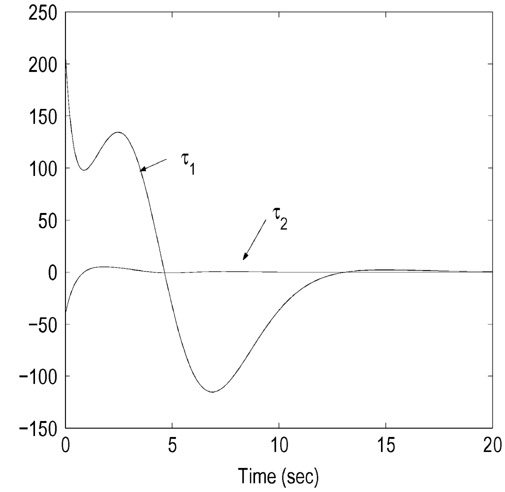

Fig. 5. Control inputs.

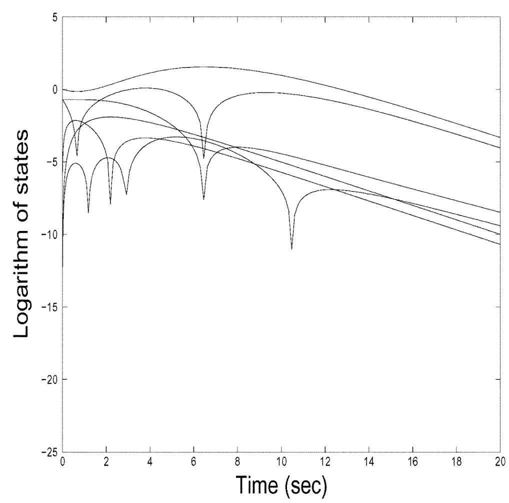

Fig. 6. Logarithm of the absolute values of each state.

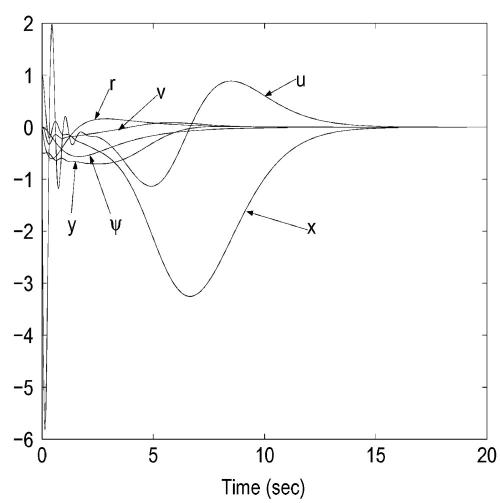

Fig. 7. Response of the state.

Fig. 8 shows that all states exponentially converge to zero because the logarithm of the absolute values of each state decays linearly, which confirms the assertion in Theorem 3. Fig. 9 shows that the control inputs are bounded and converge to zero.

By simulations, the three proposed control laws all stabilize the underactuated surface vessel to zero. Since the first control law only asymptotically stabilizes the system to the origin, the performance of

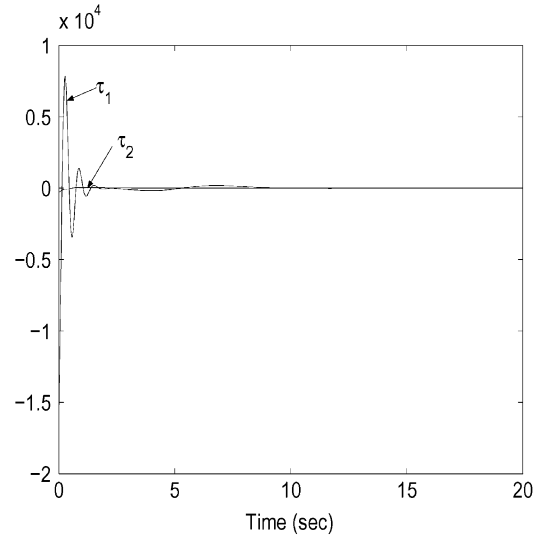

Fig. 8. Control inputs.

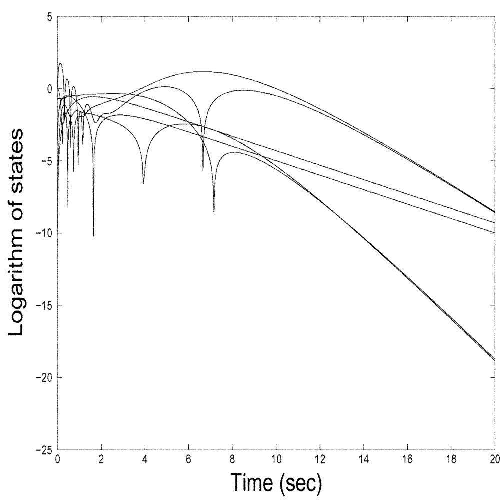

Fig. 9. Logarithm of the absolute values of each state.

the closed-loop system cannot be guaranteed. The second and the third control laws globally stabilize the state of the closed-loop system to zero with exponential convergence rates. Therefore, the performances of the closed-loop systems with exponentially stabilizing control laws are satisfactory. Moreover, with the third control law the exponential convergence rate can be assigned arbitrarily by selecting the control parameters.

VI. Conclusion

In this note, the stabilization problem of an underactuated surface vessel is considered. Three stabilizing control laws are proposed. Simulation study shows the effectiveness of the controllers. The ideas developed in this note can also be applied to design controllers of other underactuated systems.

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