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Nonlinear H_∞ Control of Uncertain Nonholonomic Systems in Chained Forms

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***Abstract*— Nonlinear H_∞ control is considered for uncertain nonholonomic systems with external disturbances in their chained form kinematic models. State feedback controllers are explicitly constructed to guarantee an $\mathcal{L}_2$ gain performance from the disturbance to the system output. Without the presence of the disturbance, the system states are regulated to the origin. Recent results in robust and adaptive control of uncertain nonholonomic systems are extended to include external disturbance inputs. A simulation example shows the effectiveness of the proposed control schemes.**

***Index Terms*— Nonlinear control, nonholonomic systems, backstepping, disturbance attenuation, $\mathcal{L}_2$ gain.**

1. Introduction

There has been increasing interest in studying uncertain robotic systems with nonholonomic constrains for the last decade due to their inherent nonlinearity and challenges in constructing analytic control solutions. A good review on the nonholonomic control systems is presented in [13], and a review on both holonomic and nonholonomic constrained systems in [19]. Among examples of nonholonomic systems are car-like robots, multi-fingered hands, space robots with linked rigid bodies, floating astronauts, and satellites with rotors. The stabilization problem was recently solved for uncertain kinematic model in its chained form in [11]. The adaptive version of a similar configuration was presented in [7]. Near optimal tracking control was designed in [18] to take into account explicitly the control effort required to solve a given problem. Combining the kinematic model with the dynamic model in a cascaded form, stabilization and tracking controllers for uncertain nonholonomic systems were designed in [6], [5], [4], [16]. It seems that the disturbance attenuation problem has not been covered much in the existing literatures. Exceptions include [3] (and some references therein), where the authors consider model reference control for perturbed nonholonomic systems with external disturbance, and an H_∞ performance is achieved at the last stage of the design.

In this paper, we consider disturbance attenuation for uncertain nonholonomic kinematic systems in their chained form models. A nonlinear H_∞ control problem is solved for such systems. Since the initial results proposed in [17], nonlinear H_∞ control has been discussed intensively for structured systems in [15], [9], [10], and constructive techniques using backstepping have been exploited. Later on, decentralized nonlinear H_∞ control was discussed in [8], [12]. Since disturbance is a commonly existing component in nonholonomic mechanic system dynamics, this paper presents new results in designing an H_∞ controller for uncertain nonholonoimc systems with external disturbance inputs. Explicit state feedback controllers are constructed so that the effect of the disturbance on the system output is attenuated to any given level in the sense of an $\mathcal{L}_2$ gain measurement. The states of the closed-loop system are regulated to the origin without the presence of disturbances. Simulation results on an example system are given to demonstrate the responses of our controlled system.

The notation used in this paper is standard. $|\cdot|$ denotes the usual Euclidean norm for vectors. We say that $z : [0, \infty) \to \Re^k$ is in $\mathcal{L}_2$ if $\int_0^\infty |z(t)|^2 dt < \infty$, and z is in $\mathcal{L}_\infty$ if $\sup_{t\geq 0} |z(t)| < \infty$. The arguments of a function are omitted sometimes in the analysis when no confusion can arise.

2. Class of Systems and Control Problem

As stated in Introduction, many nonlinear mechanical systems with nonholonomic constraints can be transformed to a canonical chained form representation. We consider a class of the uncertain nonholonomic systems in their perturbed chained form:

$$\begin{aligned}
\dot{x}_0 &= u_0 + \gamma_0(t, x_0) + p_0^T(t)\omega && (1)\\
\dot{x}_i &= u_0 x_{i+1} + \gamma_i(t, x_0, x_1, \ldots, x_i, u_0) \\
&\quad + p_i^T(t, x_0, x_1, \ldots, x_i, u_0)\omega \qquad 1 \leq i < n \\
\dot{x}_n &= u_1 + \gamma_n(t, x_0, x_1, \ldots, x_n, u_0) \\
&\quad + p_n^T(t, x_0, x_1, \ldots, x_n, u_0)\omega && (2)\\
y &= x_1 && (3)
\end{aligned}$$

where $x = (x_0, x_1, \ldots, x_n) \in \Re^{n+1}$ are the system states, $u_0, u_1 \in \Re$ are the system input, $\omega \in \Re^m$ is the disturbance input, $y \in \Re$ is the output, $\gamma_i, p_i, i = 0, 1, \ldots, n$ are unknown functions/vectors and are locally Lipschitz in states and piecewise continuous in t, and $\gamma_i(t, 0, \ldots, 0) = 0$.

Remark 1: The perturbed version of the chained form systems was recently discussed in [11], [7]. The author of [11] considers uncertain terms defined as "disturbed virtual control directions" and "input and state-driven uncertainties", and an adaptive version without the first uncertain term is discussed in [7]. The class of systems (1) and (2) considered in this paper is essentially a similar class as described in [11], [7], with external disturbance terms considered explicitly. Motivating examples of the perturbed chained form systems are given in [11].

Assumption 1: There exist positive constants k_1 and k_2 such that

$$\begin{aligned}
|\gamma_0(t, x_0)| &\leq k_1 |x_0|, \\
|p_0(t)| &\leq k_2. && (4)
\end{aligned}$$

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Assumption 2: There exist known smooth nonnegative functions $\phi_i,\ \psi_i\ (1 \le i \le n)$, and a positive constant ψ_{i0} such that

$$\begin{aligned}&|\gamma_i(t, x_0, x_1, \ldots, x_i, u_0)| \le |(x_1, \ldots, x_i)| \\ &\cdot\phi_i(x_0, x_1, \ldots, x_i, u_0), \\ &|p_i(t, x_0, x_1, \ldots, x_i, u_0)| \le |(x_1, \ldots, x_i)| \\ &\cdot\psi_i(x_0, x_1, \ldots, x_i, u_0) + \psi_{i0}. \end{aligned} \quad (5)$$

The objective of our design is to find state feedback controllers to make the closed-loop system uniformly asymptotically stable while arbitrarily attenuating the effect of the disturbance in the sense of an $\mathcal{L}_2$ gain. A precise statement of this control problem is given below:

Problem of H_∞ Almost Disturbance Decoupling: Find state feedback controllers $u_0(x_0)$ and $u_1(x)$ such that, for any given positive constant μ, the closed-loop interconnected system satisfies the following dissipation inequality

$$\int_0^\infty |y(t)|^2 dt \le \mu \int_0^\infty |\omega(t)|^2 dt + \nu(x(0)), \quad \forall \omega(t) \in \mathcal{L}_2 \quad (6)$$

where ν is a positive semi-definite function and $x(0)$ is the initial condition. Furthermore, uncertain system (2) is regulated to the origin if $\omega = 0$.

3. State Feedback Controller Design

3.1. Design u_0 for the x_0-Subsystem

We first design u_0 for the x_0-subsystem. The following lemma is used in our controller design.

Lemma 1 ([14]): Given a differentiable function $f(t) : \Re^+ \to \Re$, if $f \in \mathcal{L}_2$ and $\dot{f} \in \mathcal{L}_2$, then $f \to 0$ and $f \in \mathcal{L}_\infty$.

Lemma 2 ([14]): Given a differentiable function $f(t) : \Re^+ \to \Re$, if $f_1 \in \mathcal{L}_2$ and $f_2 \in \mathcal{L}_2$, then $f_1 + f_2 \in \mathcal{L}_2$.

Consider the following positive definite function

$$V_0 = x_0^2 + \int_t^\infty \omega^T \omega dt, \quad \forall T \ge t \quad (7)$$

Note that V_0 is not a Lyapunov function since the system may not have an equilibrium point.

Since $\omega \in \mathcal{L}_2$, there exists a constant C such that

$$\int_0^t \omega^T \omega dt + \int_t^\infty \omega^T \omega dt = C < \infty \quad (8)$$

Taking the derivative with respect to time, we obtain

$$\omega^T \omega + \frac{d}{dt}\left[\int_t^\infty \omega^T \omega dt\right] = 0 \quad (9)$$

From equations (7), (9), (1), and (4) in Assumption 1, we have

$$\begin{aligned}\dot{V}_0 &= 2x_0[u_0 + \gamma_0(t, x_0) + p_0^T(t)\omega] - \omega^T\omega \\ &\le 2x_0 u_0 + 2k_1|x_0|^2 + 2k_2|x_0||\omega| - \omega^T\omega \end{aligned} \quad (10)$$

Using the inequality

$$2ab \le a^2 + b^2,\ (a, b \in \Re) \quad (11)$$

to the third term, we get

$$\begin{aligned}\dot{V}_0 \le\ & 2x_0 u_0 + 2k_1|x_0|^2 + k_2^2|x_0|^2 \\ & + |\omega|^2 - \omega^T\omega \end{aligned} \quad (12)$$

Choose

$$u_0 = -(c_0 + k_1 + \frac{1}{2}k_2^2)x_0 \triangleq \lambda_0 x_0 \quad (13)$$

where c_0 is a positive constant. Then (12) turns to:

$$\dot{V}_0 \le -2c_0 x_0^2 \quad (14)$$

Therefore V_0 is a non-increasing function and thus $V_0 \in \mathcal{L}_\infty$ which implies that $x_0 \in \mathcal{L}_\infty$. Integrating (14), we get $x_0 \in \mathcal{L}_2$. From (4) in Assumption 1 and Lemma 2, we conclude that $\dot{x} \in \mathcal{L}_2$. Finally from Lemma 1, since $x, \dot{x} \in \mathcal{L}_2$, we get $x \to 0$.

3.2. State Scaling

From the above analysis, we see that by choosing u_0 as in (13), the x_0 state in (1) can be regulated to zero as $t \to \infty$. Since the system (2) is un-controllable in the limit $u_0 = 0$, discontinuous coordinate transformation is needed to avoid the un-controllable situation, as used in [2], [11], [7]. The following discontinuous coordinate transformation, an application of σ process [1], is defined as follows:

$$z_i = \frac{x_i}{x_0^{n-i}}, \quad 1 \le i \le n \quad (15)$$

In the new z-coordinates, the system (2) is transformed into

$$\begin{aligned}\dot{z}_i &= \lambda_0 z_{i+1} + f_i(t, x_0, z_1, \ldots, z_i) \\ &\quad + g_i(t, x_0, z_1, \ldots, z_i)\omega, \quad 1 \le i < n \\ \dot{z}_n &= u_1 + f_n(t, x_0, z_1, \ldots, z_n) \\ &\quad + g_n(t, x_0, z_1, \ldots, z_n)\omega \end{aligned} \quad (16)$$

where for $1 \le i \le n$,

$$\begin{aligned} f_i(x_0, z_1, \ldots, z_i) &= -(n-i)\lambda_0 z_i - \frac{(n-i)z_i}{x_0}\gamma_0 \\ &\quad + \frac{1}{x_0^{n-i}}\gamma_i \\ g_i(x_0, z_1, \ldots, z_i) &= -\frac{(n-i)z_i}{x_0}p_0^T + \frac{1}{x_0^{n-i}}p_i^T \end{aligned} \quad (17)$$

It is not difficult to obtain the following conditions on f_i and g_i, $1 \le i \le n$, which are analogous with those in Assumption 2:

$$\begin{aligned} |f_i(t, x_0, z_1, \ldots, z_i)| &\le |(z_1, \ldots, z_i)| \\ &\quad \bar{\phi}_i(x_0, z_1, \ldots, z_i), \\ |g_i(t, x_0, z_1, \ldots, z_i)| &\le |(z_1, \ldots, z_i)| \\ &\quad \bar{\psi}_i(x_0, z_1, \ldots, z_i) + \bar{\psi}_{i0}(x_0) \end{aligned} \quad (18)$$

where $\bar{\phi}_i$ and $\bar{\psi}_i$ are smooth nonnegative functions.

3.3. Backstepping Design for u_1

In this subsection, we apply a recursive backstepping design procedure to the dynamics (16) in the z-coordinates.

Step 1: From (16), the z_1-subsystem dynamics is:

$$\dot{z}_1 = \lambda_0 z_2 + f_1(x_0, z_1) + g_1(x_0, z_1)\omega. \quad (19)$$

To design a virtual control $z_2 = z_2^*(x_0, z_1)$ for (19), we define a storage function V_1 as

$$V_1(z_1) = z_1^2. \quad (20)$$

Taking time derivative of V_1, we get:

$$\begin{aligned}\dot{V}_1 &\leq 2z_1\lambda_0 z_2 + 2|z_1|^2\bar{\phi}_1(x_0, z_1) \\ &\quad +2|z_1|^2\bar{\psi}_1(x_0, z_1)|\omega| + 2|z_1|\bar{\psi}_{10}(x_0)|\omega| \quad (21)\end{aligned}$$

Using (11) to the last term in the above equation, we get

$$\begin{aligned}\dot{V}_1 &\leq \frac{1}{d_{11}}|z_1|^4\bar{\psi}_1^2(x_0, z_1) + \frac{1}{d_{12}}|z_1|^2\bar{\psi}_{10}^2(x_0) \\ &\quad +(d_{11} + d_{12})|\omega|^2 \quad (22)\end{aligned}$$

where d_{11} and d_{12} are positive design constants.

Choosing the virtual control z_2^* as

$$\begin{aligned}z_2^*(x_0, z_1) = \frac{1}{\lambda_0}\Big[&-\frac{c_{11}}{2}z_1 - z_1\bar{\phi}_1(x_0, z_1) \\ &-\frac{1}{2d_{11}}z_1^3\bar{\psi}_1^2(x_0, z_1) - \frac{1}{2d_{12}}z_1\bar{\psi}_{10}^2(x_0)\Big] \quad (23)\end{aligned}$$

where c_{11} is positive constant to be chosen later. Then we have

$$\dot{V}_1 \leq -c_{11}z_1^2 + d_1|\omega|^2 + 2z_1\lambda_0(z_2 - z_2^*) \quad (24)$$

where $d_1 = d_{11} + d_{12}$.

It is easy to check that $z_2^*(x_0, z_1)$ is a smooth function, and $z_2^*(x_0, 0) = 0, \frac{\partial z_2^*}{\partial x_0}(x_0, 0) = 0$.

Step 2: Augment the z_2 subsystem to the z_1-subsystem, and choose a storage function

$$V_2(z_1, z_2) = V_1(z_1) + (z_2 - z_2^*)^2 \quad (25)$$

Differentiating V_2, we get

$$\dot{V}_2 = \dot{V}_1 + 2(z_2 - z_2^*)(\lambda_0 z_3 + f_2 + g_2\omega - \dot{z}_2^*) \quad (26)$$

Note that

$$\begin{aligned}\dot{z}_2^* &= \frac{\partial z_2^*}{\partial x_0}\dot{x}_0 + \frac{\partial z_2^*}{\partial z_1}\dot{z}_1 \\ &= \frac{\partial z_2^*}{\partial z_1}\cdot\lambda_0 z_2 + \left[\frac{\partial z_2^*}{\partial x_0}\cdot(\lambda_0 x_0 + \gamma_0) + \frac{\partial z_2^*}{\partial z_1}\cdot f_1\right] \\ &\quad + \left(\frac{\partial z_2^*}{\partial x_0}\cdot p_0^T + \frac{\partial z_2^*}{\partial z_1}\cdot g_1\right)\omega \end{aligned} \quad (27)$$

Substitute (27) into (26), we get

$$\begin{aligned}\dot{V}_2 &\leq -c_{11}z_1^2 + d_1|\omega|^2 + 2(z_2 - z_2^*)\Big(\lambda_0 z_3 \\ &\quad +\lambda_0 z_1 - \frac{\partial z_2^*}{\partial z_1}\cdot\lambda_0 z_2 - \frac{\partial z_2^*}{\partial x_0}\lambda_0 x_0\Big) \\ &\quad +\Delta_2(x_0, z_1, z_2) + \eta_2(x_0, z_1, z_2)\omega \quad (28)\end{aligned}$$

where

$$\begin{aligned}\Delta_2 &= 2(z_2 - z_2^*)\left(-\frac{\partial z_2^*}{\partial x_0}\gamma_0 - \frac{\partial z_2^*}{\partial z_1}f_1 + f_2\right) \\ \eta_2 &= 2(z_2 - z_2^*)\left(-\frac{\partial z_2^*}{\partial x_0}p_0^T - \frac{\partial z_2^*}{\partial z_1}g_1 + g_2\right)\end{aligned} \quad (29)$$

Applying (4), (18), (27) and (11) to the uncertain term Δ_2, and after lengthy but simple calculations, there exist smooth nonnegative functions ϑ_{21} and ϑ_{22} such that:

$$\begin{aligned}|\Delta_2| &\leq \tilde{z}_2^2\vartheta_{21}(x_0, z_1, z_2) + l_{21}z_1^2 \\ |\eta_2\omega| &\leq \tilde{z}_2^2\vartheta_{22}(x_0, z_1, z_2) + d_{21}|\omega|^2 \quad (30)\end{aligned}$$

where l_{21}, l_{22}, and d_{21} are positive constants, $\tilde{z}_2 = z_2 - z_2^*$, and

$$\begin{aligned}&\vartheta_{21}(x_0, 0, 0) = 0, \qquad \frac{\partial\vartheta_{21}}{\partial x_0}(x_0, 0, 0) = 0. \\ &\vartheta_{22}(x_0, 0, 0) = 0, \qquad \frac{\partial\vartheta_{22}}{\partial x_0}(x_0, 0, 0) = 0. \quad (31)\end{aligned}$$

Choose the virtual control $z_3^*(x_0, z_1, z_2)$ as follows:

$$\begin{aligned}z_3^* &= -\frac{c_{22}}{2}\tilde{z}_2 - z_1 + \frac{\partial z_2^*}{\partial z_1}z_2 + \frac{\partial z_2^*}{\partial x_0}x_0 \\ &\quad -\frac{1}{2\lambda_0}\tilde{z}_2[\vartheta_{21}(x_0, z_1, z_2) + \vartheta_{22}(x_0, z_1, z_2)] \quad (32)\end{aligned}$$

where c_{22} is a positive design constant. It can be checked that

$$z_3^*(x_0, 0, 0) = 0, \qquad \frac{\partial z_3^*}{\partial x_0}(x_0, 0, 0) = 0. \quad (33)$$

Then we have

$$\begin{aligned}\dot{V}_2 &\leq -c_{21}z_1^2 - c_{22}\tilde{z}_2^2 + d_2|\omega|^2 \\ &\quad +2\lambda_0\tilde{z}_2(z_3 - z_3^*) \quad (34)\end{aligned}$$

where

$$c_{21} = c_{11} - l_{21}, \qquad d_2 = d_1 + d_{21}. \quad (35)$$

Step i ($3 \leq i \leq n-1$): Assume that from Step $i-1$, we have designed a virtual control $z_i^*(x_0, z_1, \ldots, z_{i-1})$, so that for the chosen storage function V_{i-1}, which has the property

$$z_i^*(x_0, 0, \ldots, 0) = 0 \qquad \frac{\partial z_i^*}{\partial x_0}(x_0, 0, \ldots, 0) = 0. \quad (36)$$

For the chosen storage function V_{i-1}, its time derivative is

$$\dot{V}_{i-1} \leq \sum_{j=1}^{i-1} -c_{i-1,j}\tilde{z}_j^{\,2} + d_{i-1}|\omega|^2 + 2\lambda_0\tilde{z}_{i-1}\tilde{z}_i \quad (37)$$

where $\tilde{z}_1 = z_1, \tilde{z}_j = z_j - z_j^*,\ 1 < j \leq n-1$. Choose

$$V_i = V_{i-1} + (z_i - z_i^*)^2 \quad (38)$$

Its time derivative is

$$\begin{aligned}\dot{V}_i &\leq \sum_{j=1}^{i-1} -c_{i-1,j}\tilde{z}_j^{\,2} + d_{i-1}|\omega|^2 + 2\tilde{z}_i\Bigg\{\lambda_0\tilde{z}_{i-1} \\ &\quad +\lambda_0 z_{i+1} - \frac{\partial z_i^*}{\partial x_0}\lambda_0 x_0 - \sum_{j=1}^{i-1}\frac{\partial z_i^*}{\partial z_j}\lambda_0 z_{j+1}\Bigg\} \\ &\quad +\Delta_i + \eta_i\omega \quad (39)\end{aligned}$$

where

$$\begin{aligned}\Delta_i &= 2\tilde{z}_i\left(-\frac{\partial z_i^*}{\partial x_0}\gamma_0 - \sum_{j=1}^{i-1}\frac{\partial z_i^*}{\partial z_j}f_j + f_i\right)\\ \eta_i &= 2\tilde{z}_i\left(-\frac{\partial z_i^*}{\partial x_0}p_0^T - \sum_{j=1}^{i-1}\frac{\partial z_i^*}{\partial z_j}g_j + g_i\right)\end{aligned} \tag{40}$$

Applying bounds to the two uncertain terms in the above equation, and after some calculations, we get:

$$\begin{aligned}|\Delta_i| &\leq \sum_{j=1}^{i-1} l_{i1}\tilde{z}_j^2 + \tilde{z}_i^2\vartheta_{i1}(x_0, z_1, \ldots, z_i)\\ |\eta_i\omega| &\leq \tilde{z}_i^2\vartheta_{i2}(x_0, z_1, \ldots, z_i) + d_{i1}|\omega|^2\end{aligned} \tag{41}$$

where l_{i1}, l_{i2},and d_{i1} are positive constants, ϑ_{i1} and ϑ_{i2} are smooth nonnegative functions and have the same property as in (31).

Choose virtual control as

$$\begin{aligned}z_{i+1}^* &= -\frac{c_{ii}}{2}\tilde{z}_i - z_{i-1} + \frac{\partial z_i^*}{\partial x_0}x_0 + \sum_{j=1}^{i-1}\frac{\partial z_i^*}{\partial z_j}z_{j+1}\\ &\quad -\frac{1}{2\lambda_0}\tilde{z}_i[\vartheta_{i1} + \vartheta_{i2}]\end{aligned} \tag{42}$$

Substitute (41) and (42) into (39), we obtain:

$$\dot{V}_i \leq \sum_{j=1}^{i} -c_{ij}\tilde{z}_j^{\,2} + d_i|\omega|^2 + 2\lambda_0\tilde{z}_i\tilde{z}_{i+1} \tag{43}$$

where

$$c_{ij} = c_{i-1,j} - l_{i1},\ j < i \quad d_i = d_{i-1} + d_{i1}$$

Step n: In this step, we design the true control u_1 for the whole system. Choose the storage function

$$V_n = V_{n-1} + \tilde{z}_n^2 \tag{44}$$

where $\tilde{z}_n = z_n - z_n^*$.

Its time derivative along the system dynamics (16) is

$$\begin{aligned}\dot{V}_n &\leq \sum_{j=1}^{n-1} -c_{n-1,j}\tilde{z}_j^{\,2} + d_{n-1}|\omega|^2 + 2\tilde{z}_n\left\{\lambda_0\tilde{z}_{n-1}\right.\\ &\quad\left. +u_1 - \frac{\partial z_n^*}{\partial x_0}\lambda_0 x_0 - \sum_{j=1}^{n-1}\frac{\partial z_n^*}{\partial z_j}\lambda_0 z_{j+1}\right\}\\ &\quad +\Delta_n + \eta_n\omega\end{aligned} \tag{45}$$

where

$$\begin{aligned}\Delta_n &= 2\tilde{z}_n\left(-\frac{\partial z_n^*}{\partial x_0}\gamma_0 - \sum_{j=1}^{n-1}\frac{\partial z_n^*}{\partial z_j}f_j + f_n\right)\\ \eta_n &= 2\tilde{z}_n\left(-\frac{\partial z_n^*}{\partial x_0}p_0^T - \sum_{j=1}^{n-1}\frac{\partial z_n^*}{\partial z_j}g_j + g_n\right)\end{aligned} \tag{46}$$

As in the previous steps, we can find smooth nonnegative functions ϑ_{n1} and ϑ_{n2} such that:

$$\begin{aligned}|\Delta_n| &\leq \sum_{j=1}^{n-1} l_{n1}\tilde{z}_j^2 + \tilde{z}_n^2\vartheta_{n1}(x_0, z_1, \ldots, z_n)\\ |\eta_n\omega| &\leq \tilde{z}_n^2\vartheta_{n2}(x_0, z_1, \ldots, z_n) + d_{n1}|\omega|^2\end{aligned} \tag{47}$$

where l_{n1}, l_{n2}, and d_{n1} are positive constants.

Our true control u_1 is designed to be:

$$\begin{aligned}u_1 &= -\frac{c_{nn}}{2}\tilde{z}_n - \lambda_0 z_{n-1} + \lambda_0\frac{\partial z_n^*}{\partial x_0}x_0\\ &\quad +\lambda_0\sum_{j=1}^{n-1}\frac{\partial z_n^*}{\partial z_j}z_{j+1} - \frac{1}{2}\tilde{z}_n[\vartheta_{n1} + \vartheta_{n2}]\end{aligned} \tag{48}$$

Substitute (47) and (48) into (45), we obtain:

$$\dot{V}_n \leq \sum_{j=1}^{n} -c_{nj}\tilde{z}_j^{\,2} + d_n|\omega|^2 \tag{49}$$

where

$$c_{nj} = c_{n-1,j} - l_{n1},\ j < n \ \ d_n = d_{n-1} + d_{n1}$$

3.4. Switching Strategy

In the case of $x_0 = 0$, we need to design a switching controller to avoid singularity of the state scaling (15). It has been pointed out in [11], [7] that for uncertain terms that do not satisfy the Lipschitz condition, the states may blow up within a finite time. Following the similar ideas, we choose a different u_0 when $x_0 = 0$:

$$u_0 = \lambda_0 x_0 + \bar{C} \tag{50}$$

where $\bar{C}$ is any positive constant. Applying it to the time derivative of V_0 in (7), we obtain

$$\dot{V}_0 \leq -2c_0x_0^2 + \bar{C}x_0, \tag{51}$$

from which we can conclude the boundedness of x_0.

For the stability of states $x_1, \ldots$, and x_n, replace u_0 by (50) (instead of (13)) in the system dynamics (2). Then applying the same backstepping design procedure to the new z-coordinate dynamics as described above, we get an essentially same inequality as (49), which implies that states $x_1, \ldots$, and x_n do not blow up.

3.5. Main Theorem

We are now ready to present our main result:

Theorem 1: For the system (1)-(3) under Assumptions 1 and 2, the control laws (13) and (48) and the switching strategy presented above solve the Problem of H_∞ Almost Disturbance Decoupling.

Proof: We know that the state x_0 is regulated to 0 as $t \to \infty$ from Section 3.1. In the z-coordinates, from the last step in the recursive backstepping design, we obtained (49). If we choose design parameters l_{i1}, l_{i2}, and $c_{ij},\ 1 \leq i, j \leq n, j \leq i$ such that $c_{nj} > 0$, when $\omega = 0$, we have a positive definite and radially unbounded Lyapunov function (44), and its time derivative is negative definite. Therefore, we can conclude that

$\widetilde{z}_i,\ 1 \le i \le n$ are uniformly asymptotically stable (UAS). Since $\widetilde{z}_i = z_i - z_i^*$, and $z_i^*(x_0, 0, \ldots, 0) = 0$, we get that z_i is UAS, which implies UAS of states x in the original coordinates.

When $\omega \neq 0$, taking the integral of (49) along time t, we can obtain

$$\int_0^\infty |y(t)|^2 dt \le \mu \int_0^\infty |\omega(t)|^2 dt + \nu(\widetilde{z}(0)) \qquad (52)$$

where

$$\mu = d_n / c_{n1} \ \ \nu(\widetilde{z}(0)) = \sum_{i=1}^{n} \left(\widetilde{z}_i(0)\right)^2 .$$

This complete the proof of Theorem 1.

4. A Simulation Example

We consider the following example system which belongs to the class of systems in this paper's interest:

$$\begin{aligned} \dot{x}_0 &= u_0 + 0.1\omega \\ \dot{x}_1 &= x_2 u_0 + d_1(t) x_1^2 + d_2(t)\omega \\ \dot{x}_2 &= u_1 + d_3(t) x_1 \omega \end{aligned} \qquad (53)$$

where

$$d_1(t) = 0.1\sin(t),\ d_2(t) = 0.1\cos(0.5t),$$

$$d_3(t) = 0.2,\ \omega = e^{-t}.$$

We choose $u_0 = -x_0$ for the x_0-subsystem, and apply the state scaling

$$z_1 = \frac{x_1}{x_0},\ z_2 = x_2.$$

Following the design procedure showed in Steps 1 and 2 of Section 3.3, we have

$$z_2^* = 2z_1 + z_1 \mathrm{a}bs(z_1) x_0 d_{1m} + z_1 d_{2m}^2 / x_0^2 + z_1^3 / x_0^2,$$

$$\widetilde{z}_2 = z_2 - z_2^*,$$

$$a_1(x_0, z_1) = 2 + d_{1m} \mathrm{a}bs(x_0) 2 z_1 \mathrm{s}gn(z_1) + \frac{d_{2m}^2}{x_0^2} + 3\frac{z_1^2}{x_0^2},$$

$$a_2(x_0, z_1) = d_{1m} z_1 \mathrm{a}bs(z_1) \mathrm{s}gn(x_0) - 2d_{2m}^2 \frac{z_1}{x_0^3} - d_{2m}^2 \frac{z_1^3}{x_0^3},$$

$$\begin{aligned} u_1 = & -\widetilde{z}_2 + 2z_1 + a_1(z_1 - z_2) + a_2(-x_0) \\ & -\widetilde{z}_2 a_1^2 d_{1m}^2 x_0^4 z_1^2 - 0.5\widetilde{z}_2 a_2^2 - 0.5\widetilde{z}_2 a_1^2 d_{2m}^2 \\ & -0.5\widetilde{z}_2 d_3^2 x_0^2 z_1^2 \end{aligned} \qquad (54)$$

where d_{1m}, d_{2m}, and d_3 are the maximum values of $d_1(t), d_2(t)$,and $d_3(t)$, respectively, $\mathrm{s}gn(x)$ is the sign of x, and $\mathrm{a}bs(x)$ is the absolute value of x. Note that the selection of design parameters affects the performance of the closed-loop system but not the stability.

The responses of the closed-loop system and the time history of control inputs are shown in Figure 1. It can be seen that our control scheme achieves satisfactory performances.

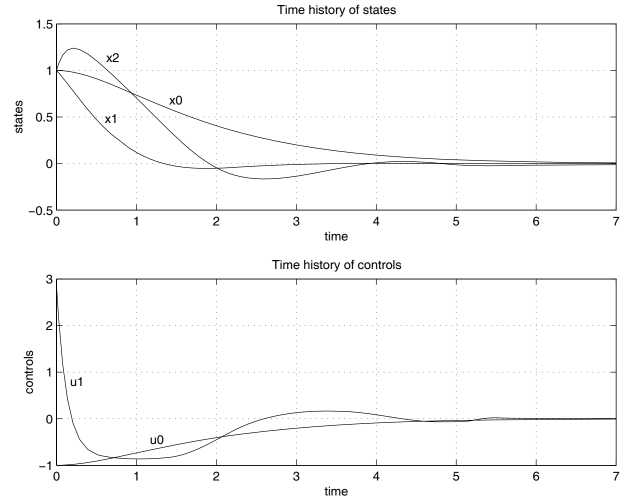

Fig. 1. Closed-loop responses and control input history.

5. Conclusions

In this paper, we consider nonlinear H_∞ control for a class of uncertain nonholonomic systems in their chained forms. Recent results presented in [11], [7] are extended to the class of uncertain chained form systems with external disturbances. Constructive controllers are designed using backstepping, and the so-called H_∞ almost disturbance decoupling problem is solved. The states of the closed-loop system are regulated to the origin without the presence of disturbances. With the presence of the disturbance, the effect of the disturbance on the system output is attenuated to any given level in the sense of an $\mathcal{L}_2$ gain measurement. A simulation example shows the effectiveness of the proposed control scheme. Future research includes applying the proposed disturbance attenuation design to practical uncertain nonholonomic systems such as car-like robots, and other under-actuated mechanical systems.

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