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Average consensus with weighting matrix design for quantized communication on directed switching graphs

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SUMMARY

We study average consensus for directed graphs with quantized communication under fixed and switching topologies. In the presence of quantization errors, conventional consensus algorithms fail to converge and may suffer from an unbounded asymptotic mean square error. We develop robust consensus algorithms to reduce the effect of quantization. Specifically, we introduce a robust weighting matrix design and use the H_∞ performance index to measure the sensitivity from the quantization error to the consensus deviation. Linear matrix inequalities are used as design tools. The mean square deviation is proven to converge and its upper bound is explicitly given in the case of fixed topology with probabilistic quantization. Numerical results demonstrate the effectiveness of this method. Copyright © 2012 John Wiley & Sons, Ltd.

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KEY WORDS: average consensus; quantization; switching directed graph; linear matrix inequalities; H_∞ norm

1. INTRODUCTION

In recent years, distributed average consensus and gossiping algorithms have received considerable attention in multiple agent coordination and distributed sensor networks including distributed estimation [1–3] and cooperative control [4–7]. In most of these studies, they assume that each sensor, say in a wireless sensor network, can communicate with its neighbors without any distortion. This assumption, however, may not be true in practice when considering quantization brought by digital communication. As pointed out in [8], in the presence of errors or noise, the conventional linear consensus algorithm diverge and may have an unbounded asymptotic mean square error. Motivated by this challenge, average consensus with quantized communication has attracted a lot of studies over the past few years.

On the basis of the convergence of controlled Markov processes, a protocol for undirected topologies is introduced in [9] to deal with the distributed consensus problem in the presence of quantization errors. This protocol adds a controlled amount of statistical dither before quantization and can reach almost sure convergence to a finite random variable. However, there is no guarantee that this finite random variable is equal to the average of the nodes' initial values. Quantized consensus studied in [10, 11] restrict the value of each node to be an integer and reaches average consensus in a 'quantized consensus' sense, which means that every node takes one of two neighboring quantization values while preserving the average. However, this is not a strict consensus and nodes reach the quantized average instead of the true average. Average consensus can be asymptotically achieved with the proposed encoding–decoding scheme in [12], which is based on quantization of scaled innovations. However, this method requires that all nodes are synchronized, which may not be true

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in a practical network. The authors in [13] consider the continuous-time quantized consensus problem, where a discontinuous protocol is designed to restrict the dynamic evolving towards the initial average. However, there exists unavoidable chattering in practice, and replacing the discontinuous function with a hysteresis, one remedies the problem but reduces accuracy. A protocol featuring an attractive property of preserving the initial average throughout the iterations was proposed in [14]. This protocol drives the system close to the average consensus with a bounded error. To further reduce the quantization error effects, a sequence averaging algorithm based on this sum-preserving protocol was proposed in [15]. Meanwhile, [16–18] studied the quantization problem in gossip based consensus algorithms. In [18], the author proposed two different protocols, that is, a globally quantized strategy and a partially quantized strategy, respectively. For the globally quantized strategy, it was shown that all states eventually reach agreement, but the value of the agreement is different from the initial average. In contrast, the partially quantized strategy preserves the average and the consensus error is bounded by one (states are quantized to integers) after a finite period of time. The partially quantized strategy is also studied in [16, 17] and similar conclusions are extended to general quantizers. It was shown that the consensus error of this algorithm is bounded by one but whether or not the consensus error can be further reduced was not explored.

In this paper, we consider distributed consensus on directed switching communication graphs, which are used to model many real communication systems [19]. Specifically, inspired by the idea of robust H_∞ consensus control [20], we model the problem as a robust consensus design problem with mismatched quantization noises and design the weighting matrix of the average consensus protocol introduced in [14] under the H_∞ performance index to mitigate the consensus deviation caused by quantization errors. Using linear matrix inequality (LMI) as design tools, we show that the consensus deviation is bounded by a pre-designated constant. For a probabilistic quantization scheme on directed graphs, the covariance matrix of states is proven to converge and its steady state value is given. Moreover, the mean square deviation from consensus is proved to be upper bounded and the upper bound is explicitly given in terms of the quantization standard deviation and the design parameter. Extensive simulations show the proposed algorithm over-performs existing ones in reducing consensus deviation.

The paper has the following contributions. First, most studies on average consensus with quantization are limited to undirected graphs [9, 10, 12, 13, 15–18] without allowing arbitrary switching [9–11, 14, 15]. Our results are more general for both fixed and switching directed graph with quantization, and an undirected graph is treated as a special case of directed graphs. Second, compared with the work [21–23], where weighting matrix design is considered to optimize the convergence rate without a quantizer, we aim at reducing consensus deviation caused by quantization errors, and the upper bound of the mean square derivation is given for a fixed directed graph under a probabilistic quantization scheme. Third, compared with the work of [20], which studies H_∞ consensus control in continuous time with an additive noise satisfying matching conditions, the discrete-time quantized consensus problem is studied in this paper with a mismatched quantization noise.

Notations and symbols: $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and complex numbers, respectively, $\mathrm{diag}([\mathbf{A_1} \ \dots \ \mathbf{A_n}])$ denotes the block diagonal matrix with $\mathbf{A}_i$ as the ith element, $\mathrm{eig}(\mathbf{A})$ represents the eigenvalue of $\mathbf{A}$, $\mathbf{A}^T$ represents the transpose of $\mathbf{A}$, $\mathrm{trace}(\mathbf{A})$ represents the trace of $\mathbf{A}$, $\mathbf{I}$ represents the identity matrix, $\mathrm{vec}(\mathbf{A})$ means the vectorization of a matrix $\mathbf{A}$, $\sup(x)$ denotes the supremum of x, $\mathbf{1}$ represents the vector with all entries equal to 1, $\|\mathbf{A}\|_\infty$ is the infinity norm of a matrix $\mathbf{A}$, $\|\mathbf{A}\|$ is the 2-norm of a matrix $\mathbf{A}$, $\otimes$ is the Kronecker product, $\mathbf{A} < 0$ means $\mathbf{A}$ is negative definite, and $|a|$ represents the absolute value of $a \in \mathbb{C}$.

2. PRELIMINARIES

In this section, we list some preliminaries on graph theory and quantization.

2.1. Graph theory

A directed graph $G(V, E, \mathbf{A})$ is denoted by $(V, E, \mathbf{A})$, where V is the set of nodes, E is the set of edges with $E \subseteq V \times V$, and $\mathbf{A} = [a_{ij}]$ is the weighted adjacency matrix. The in-degree

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and out-degree of a node in the directed graph is defined as $\deg_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ji}$ and $\deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ij}$, respectively. The directed graph G is said to be balanced if the in-degree equals to the out-degree for each node in the graph. A special case of the balanced graph is the undirected graph, which bears the property of $a_{ji} = a_{ij}$ for all i, j. A directed graph G is called strongly connected if there always exists a sequence of consecutive edges starting from a given node i to another given node j, where node i and node j could be any node in the graph only if $i \neq j$. The degree matrix $\mathbf{\Delta} = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ij} = 0$ for all $i \neq j$ and $\Delta_{ii} = \deg_{\text{out}}(v_i)$ for all i. The Laplacian matrix $\mathbf{L}$ of the graph G is defined as $\mathbf{L} = \mathbf{\Delta} - \mathbf{A}$.

In a continuous-time linear consensus protocol, $-\mathbf{L}$ works as the system matrix and the system converges to the initial average under the condition that the graph G is balanced and strongly connected. In the corresponding discrete-time linear consensus protocol, a stochastic matrix $\mathbf{W} = [w_{ij}]$ works as the system matrix (a matrix is called a stochastic matrix if its row sums are 1 and each entry of the matrix is non-negative). For a balanced graph, the transpose of its Laplacian matrix is also a Laplacian matrix. Correspondingly, for a balanced graph G, the transpose of its stochastic matrix is also a stochastic matrix. In matrix form, $\mathbf{1}^T\mathbf{W} = \mathbf{1}^T$. The matrix, if both itself and its transpose are stochastic matrices, is called a doubly stochastic matrix. For a strongly connected graph, the rank of its Laplacian matrix is equal to $(n-1)$, where n is the dimension of the Laplacian matrix $\mathbf{L}$. Correspondingly, the rank of the stochastic matrix is also equal to $(n-1)$ for a strongly connected graph.

A frequently used weighting matrix $\mathbf{M}$ is obtained by using maximum-degree weights [2, 14]. It is defined as

$$m_{ij} = \begin{cases} \frac{1}{d_M+1} & \text{if } \{i, j\} \in E \\ 1 - \frac{\deg_{\text{out}}(v_i)}{d_M+1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where d_M is the maximal out-degree of the graph.

2.2. *Quantization schemes*

There are two types of quantization schemes utilized in existing literatures, that is, the deterministic quantization scheme [14, 24] and the probabilistic quantization scheme [15]. Suppose the range and bits for quantization are $[-\eta, \eta]$ and b, respectively. Thus, we obtain $2^b - 1$ uniform intervals with a step of $\delta = 2\eta/(2^b - 1)$. The deterministic quantization rounds a real number x to $Q(x)$ according to

$$Q(x) = -\eta + i\delta$$
$$\text{if } x \in [-\eta + (i - 0.5)\delta, -\eta + (i + 0.5)\delta)$$

where i is an integer. The probabilistic quantization rounds a real number $x \in [-\eta + i\delta, -\eta + (i+1)\delta]$ in a probabilistic manner as follows [15]

$$\begin{cases} P(Q(x) = -\eta + i\delta) = 1 - r \\ P(Q(x) = -\eta + (i+1)\delta) = r \end{cases}$$

where $r = (x + \eta - i\delta)/\delta$. As argued in [15], the quantization error introduced by the probabilistic quantization scheme is a random variable with zero mean and a finite variance.

2.3. *L_{2e} space and finite L_2 gain*

For a discrete-time sequence $\mathbf{x} = (x_0, x_1, x_2, ...)$, its L_2 norm is defined as

$$\|\mathbf{x}\|_{L_2} \triangleq \left(\sum_{i=0}^{\infty} |x_i|^2\right)^{\frac{1}{2}} \tag{1}$$

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The space $L_2[0,+\infty)$ is defined as the set of all discrete-time sequences $\mathbf{x}$ such that $\|\mathbf{x}\|_{L_2} < +\infty$. From a system point of view [25], if we think $\mathbf{x} \in L_2[0,+\infty)$ as a 'well-behaved' input, the question to ask is whether the output $\mathbf{y}$ will be 'well-behaved' in the sense that $\mathbf{y} \in L_2[0,+\infty)$. To deal with unbounded 'ever-growing' signals, an extended space $L_{2e}[0,\tau)$ is defined as the following

$$L_{2e} = \{\mathbf{x}|\mathbf{x}_\tau \in L_2, \forall \tau \in [0,\infty)\} \tag{2}$$

where $\mathbf{x}_\tau$ is a truncation of the sequence $\mathbf{x}$ defined by

$$\mathbf{x}_\tau(t) = \begin{cases} \mathbf{x}(t) & 0 \leqslant t \leqslant \tau \\ 0 & t > \tau \end{cases}$$

The extended space L_{2e} is a linear space that contains the unextended space L_2 as a subset. A mapping $H : L_{2e}[0,\tau) \to L_{2e}[0,\tau)$ is said to have a finite L_2 gain if there exists finite constants γ and β such that

$$\|(H(\mathbf{x}))_\tau\|_{L_2} \leqslant \gamma \|\mathbf{x}_\tau\|_{L_2} + \beta \tag{3}$$

for all $\mathbf{x} \in L_{2e}[0,\tau)$ and $\tau \in [0,+\infty)$ [25, 26].

3. AVERAGE PRESERVING CONSENSUS WITH QUANTIZATION

In this paper, we study the following consensus protocol with quantization

$$x_i(t+1) = x_i(t) + \sum_{j \in N_i} w_{ji}(Q(x_j(t)) - Q(x_i(t))) \tag{4}$$

where $\mathbf{W} = [w_{ij}]$ is a doubly stochastic matrix, N_i is the neighborhood of node i and $Q(\cdot)$ is the quantization function. We can write (4) in a matrix equation form:

$$\mathbf{x}(t+1) = \mathbf{W}\mathbf{x}(t) - \mathbf{L}(Q(\mathbf{x}(t)) - \mathbf{x}(t)) \tag{5}$$

where $\mathbf{L} = \mathbf{I} - \mathbf{W}$. This protocol was first proposed in [14] and independently studied in [15] because of its attractive property in the fields, such as signal processing and data fusion, where the average of state values is invariant with time. This property can be easily verified by multiplying $\mathbf{1}^T$ on both sides of (5).

The presence of quantization function $Q(\cdot)$ in this protocol introduces nonlinearity into the linear consensus model, which makes the new nonlinear model hard to analyze directly as compared with the following conventional consensus protocol,

$$\mathbf{x}(t+1) = \mathbf{W}\mathbf{x}(t) \tag{6}$$

The difference lies in that the protocol (5) introduces an extra nonlinear item $Q(\mathbf{x}(t)) - \mathbf{x}(t)$. One way to study the new system is to treat this extra item as a disturbance to the nominal linear system (6), as shown in the following. This allows us to utilize some tools for linear consensus protocol. Henceforth, we write (5) as

$$\mathbf{x}(t+1) = \mathbf{W}\mathbf{x}(t) - \mathbf{L}\mathbf{v}(t) \tag{7}$$

where $\mathbf{v}(t) = Q(\mathbf{x}(t)) - \mathbf{x}(t)$ is the disturbance.

Remark 1

Note that the doubly stochastic matrix $\mathbf{W}$ in (4) corresponds to an undirected or a directed balanced graph. However, for a given general directed graph, we can construct $\mathbf{W}$ so that it is doubly stochastic, which is carried out by imposing the constraint $\mathbf{W}\mathbf{1} = \mathbf{1}$ and $\mathbf{W}^T\mathbf{1} = \mathbf{1}$ to form a balanced graph. That is, define an $n \times n$ matrix $\mathbf{W} = [w_{ij}]$ with $w_{kl} = 0$ if there is no edge from the kth node to the lth one, and then solve the equality constraint $\mathbf{W}\mathbf{1} = \mathbf{1}$ and $\mathbf{W}^T\mathbf{1} = \mathbf{1}$ to obtained a doubly stochastic $\mathbf{W}$ in terms of $p - 2n$ independent scalar variables, where p is the number of directed edges on the graph [21].

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Remark 2
The quantized consensus protocol (4) uses a simple quantization scheme to achieve average consensus and are applicable to both the deterministic and probabilistic quantization. Other types of consensus protocols with quantization exist in literatures, such as [9, 12]. The quantization scheme in [9] intentionally introduced statistical dither and can reach almost sure convergence to a finite random variable. But the convergence value may not be the average of the nodes' initial values, which indicates that the consensus is reached but not the average consensus. The dynamic encoding–decoding scheme in [12] is based on quantization of scaled innovations and requires that all nodes are strictly synchronized to reach ideal average consensus. It works for undirected graphs only. Comparing with these existing quantizers, the quantized consensus protocol (4) uses simple quantizers, as those described in Section 2.2, and are applicable to both undirected and directed graphs.

4. PROBLEM FORMULATION

It is the quantization error in protocol (4) that leads to the deviation from the average consensus. This motivates us to utilize a robust control strategy to reduce the effect of the quantization error.

Representing the deviation $\mathbf{x}(t) - \frac{\mathbf{1}\mathbf{1}^T\mathbf{x}(0)}{n}$ by $\mathbf{z}(t)$, we have

$$\begin{aligned}\mathbf{x}(t+1) &= \mathbf{W}\mathbf{x}(t) - \mathbf{L}\mathbf{v}(t)\\ \mathbf{z}(t+1) &= \mathbf{x}(t+1) - \frac{\mathbf{1}\mathbf{1}^T\mathbf{x}(0)}{n}\end{aligned} \tag{8}$$

Consider the following H_∞ norm objective function

$$\|\mathbf{G}_{vz}\|_\infty < \gamma \tag{9}$$

where $\mathbf{G}_{vz}$ is the transfer function from $\mathbf{v}$ to $\mathbf{z}$ of (8), $\|\mathbf{G}_{vz}\|_\infty$ is the H_∞ norm of $\mathbf{G}_{vz}$ and γ is a positive constant.

The H_∞ norm performance index (9) is defined in the complex frequency-domain. It corresponds to the L_2 gain in the time domain. This inequality imposes a constraint on the L_2 gain from the disturbance $\mathbf{v}(t)$ to $\mathbf{z}(t)$, which measures the deviation from the average consensus. That is to say, (9) is equivalent to the following time domain inequality

$$\sup_{t>0} \sum_{\tau=0}^{t} \left(\mathbf{z}^T(\tau)\mathbf{z}(\tau) - \gamma^2\mathbf{v}^T(\tau)\mathbf{v}(\tau)\right) < \text{constant} \tag{10}$$

where the constant on the right side of the inequality is due to the effect of the initial states to the output $\mathbf{z}$ and is determined by the initial states.

Problem Statement: Design the weighting matrix $\mathbf{W}$ of system (8) to achieve a bounded consensus error of $\mathbf{x}(t)$ and to satisfy the performance index (10) (equivalent to the performance index (9)).

Remark 3
Note that the matrix $\mathbf{W}$ is a stochastic matrix. The traditional robust H_∞ control theory is invalid because $\mathbf{W}$ has an eigenvalue of 1. This point is analogous to the continuous-time case with a singular Laplacian matrix as the system matrix, to which traditional H_∞ control fails [6, 20].

5. MAIN RESULTS

In this section, we present our main results on the weighting matrix design. We use a linear transformation to transform the original consensus problem to an equivalent stabilization problem. This transformation simplifies the analysis. On the basis of this transformation, we use a set of LMIs to design the weighting matrices for fixed graphs and switching graphs, respectively. In addition, we study $E(\mathbf{z}(t)\mathbf{z}^T(t))$, where $E(\cdot)$ denotes the expected value, for the protocol on a fixed graph with the probabilistic quantization scheme, which reveals the statistical property of this method.

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5.1. State transformation

We have the following proposition about the state transformation.

Proposition 1
For a strongly connected, balanced and directed graph, system (8) is equivalent to the following decoupled system:

$$\begin{aligned} \mathbf{y}_0(t+1) &= \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\mathbf{y}_0(t) - \mathbf{T}_0\mathbf{L}\mathbf{v}(t) \\ y_a(t+1) &= y_a(t) \\ \mathbf{z}(t+1) &= \mathbf{T}_0^T\mathbf{y}_0(t+1) \end{aligned} \tag{11}$$

under the similarity transformation

$$\begin{bmatrix} \mathbf{y}_0(t) \\ y_a(t) \end{bmatrix} = \mathbf{y}(t) = \mathbf{T}\mathbf{x}(t) \tag{12}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_0 \\ \frac{\mathbf{1}^T}{\sqrt{n}} \end{bmatrix} \tag{13}$$

and satisfies $\mathbf{T}^T\mathbf{T} = \mathbf{T}\mathbf{T}^T = \mathbf{I}$, $\mathbf{T}_0$ is a $(n-1)\times n$ matrix.

Proof
Because

$$\mathbf{I} = \mathbf{T}\mathbf{T}^T = \begin{bmatrix} \mathbf{T}_0\mathbf{T}_0^T & \mathbf{T}_0\frac{\mathbf{1}}{\sqrt{n}} \\ \frac{\mathbf{1}^T}{\sqrt{n}}\mathbf{T}_0^T & 1 \end{bmatrix} = \mathbf{T}^T\mathbf{T} = \mathbf{T}_0^T\mathbf{T}_0 + \frac{\mathbf{1}\mathbf{1}^T}{n} \tag{14}$$

we have,

$$\mathbf{T}_0\mathbf{1} = \mathbf{0}, \mathbf{T}_0\mathbf{T}_0^T = \mathbf{I}, \mathbf{T}_0^T\mathbf{T}_0 = \mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{n} \tag{15}$$

Representing the system (7) by the new variable $\mathbf{y}(t)$, we have

$$\begin{aligned} \mathbf{y}(t+1) &= \mathbf{T}\mathbf{W}\mathbf{T}^T\mathbf{y}(t) - \mathbf{T}\mathbf{L}\mathbf{v}(t) \\ &= \begin{bmatrix} \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T & \mathbf{T}_0\mathbf{W}\frac{\mathbf{1}}{\sqrt{n}} \\ \frac{\mathbf{1}^T}{\sqrt{n}}\mathbf{W}\mathbf{T}_0^T & \frac{\mathbf{1}^T}{\sqrt{n}}\mathbf{W}\frac{\mathbf{1}}{\sqrt{n}} \end{bmatrix}\mathbf{y}(t) - \begin{bmatrix} \mathbf{T}_0\mathbf{L}\mathbf{v}(t) \\ \frac{\mathbf{1}^T}{\sqrt{n}}\mathbf{L}\mathbf{v}(t) \end{bmatrix} \end{aligned} \tag{16}$$

For the strongly connected and balanced graph, $\mathbf{W}$ is a doubly stochastic matrix. Therefore, $\mathbf{W}\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T\mathbf{W} = \mathbf{1}^T$. Because $\mathbf{L} = \mathbf{I} - \mathbf{W}$, we have $\mathbf{1}^T\mathbf{L} = \mathbf{1}^T - \mathbf{1}^T\mathbf{W} = \mathbf{0}$. Together with (15), we obtain

$$\mathbf{y}(t+1) = \begin{bmatrix} \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}\mathbf{y}(t) - \begin{bmatrix} \mathbf{T}_0\mathbf{L}\mathbf{v}(t) \\ 0 \end{bmatrix} \tag{17}$$

That is,

$$\begin{aligned} \mathbf{y}_0(t+1) &= \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\mathbf{y}_0(t) - \mathbf{T}_0\mathbf{L}\mathbf{v}(t) \\ \mathbf{y}_a(t+1) &= \mathbf{y}_a(t) \end{aligned} \tag{18}$$

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As to the output $\mathbf{z}(t)$, we can represent it in terms of $\mathbf{y}_0(t)$

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{x}(t) - \frac{\mathbf{1}\mathbf{1}^T\mathbf{x}(0)}{n} \\ &= \mathbf{x}(t) - \frac{\mathbf{1}\mathbf{1}^T\mathbf{x}(t)}{n} \\ &= \mathbf{T}_0^T\mathbf{T}_0\mathbf{x}(t) \\ &= \mathbf{T}_0^T\mathbf{y}_0(t)\end{aligned} \tag{19}$$

The transformation matrix $\mathbf{T}$ is orthogonal, so the new system with $\mathbf{y}$ as the state is equivalent to the original one. This completes the proof. □

Remark 4
The output $\mathbf{z}(t)$ only depends on $\mathbf{y}_0(t)$, which is not coupled with $y_a(t)$. Therefore, we only need to study the dynamics of $\mathbf{y}_0(t)$ in order to study the output $\mathbf{z}(t)$.

Remark 5
Matrix $\mathbf{T}$ is not unique. One example is the matrix with all rows equal to the normalized eigenvectors of matrix $\mathbf{1}\mathbf{1}^T$ with the eigenvector $\frac{\mathbf{1}^T}{\sqrt{n}}$ in the last row.

Remark 6
The dynamic of $\mathbf{y}_0(t)$, which completely determines $\mathbf{z}(t)$, is asymptotically stable because all the eigenvalues of the system matrix $\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T$ locates inside the unit circle. This can be observed from the following analysis,

$$\mathbf{T}\mathbf{W}\mathbf{T}^T = \begin{bmatrix} \mathbf{T}_0 \\ \frac{\mathbf{1}^T}{\sqrt{n}} \end{bmatrix} \mathbf{W} \begin{bmatrix} \mathbf{T}_0^T & \frac{\mathbf{1}}{\sqrt{n}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

Note that $\mathbf{T}$ is an orthogonal matrix, so $\mathbf{W}$ and $\begin{bmatrix} \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$ are similar matrices and they have the same eigenvalues. For a strongly connected graph, the associated stochastic matrix has one eigenvalue equal to 1 and the magnitude of all the other eigenvalues is less than 1. Therefore, we conclude the eigenvalues of $\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T$ have magnitude less than 1. As a result, the system with $\mathbf{v}(t)$ as the input and $\mathbf{z}(t)$ as the output is bounded-input bounded-output. Therefore, we conclude the output $\mathbf{z}(t)$ is bounded because of the boundedness of the disturbance $\mathbf{v}(t)$.

5.2. Directed graph with fixed topology

The following lemma gives a sufficient condition to satisfy the design objective (10).

Lemma 1
For a strongly connected, balanced and directed graph with a fixed topology, the system (8) meets the objective function (10) if there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfying:

$$\begin{bmatrix} \mathbf{I} + \mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T - \mathbf{P} & -\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L} \\ -\mathbf{L}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T & -\gamma^2\mathbf{I} + \mathbf{L}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L} \end{bmatrix} < 0 \tag{20}$$

Proof
Define the Lyapunov function as follows:

$$V(t) = \mathbf{y}_0^T(t)\mathbf{P}\mathbf{y}_0(t) \tag{21}$$

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where $\mathbf{P}$ is a symmetric positive definite matrix. Denote

$$\begin{aligned} J(t) &= \sum_{\tau=0}^{t} \mathbf{z}^T(\tau)\mathbf{z}(\tau) - \gamma^2 \mathbf{v}^T(\tau)\mathbf{v}(\tau) \\ &= \sum_{\tau=0}^{t} \mathbf{y}_0^T(\tau)\mathbf{T}_0\mathbf{T}_0^T\mathbf{y}_0(\tau) - \gamma^2 \mathbf{v}^T(\tau)\mathbf{v}(\tau) \\ &= \sum_{\tau=0}^{t} \mathbf{y}_0^T(\tau)\mathbf{y}_0(\tau) - \gamma^2 \mathbf{v}^T(\tau)\mathbf{v}(\tau) \end{aligned} \tag{22}$$

Note that

$$\begin{aligned} V(t+1) - V(0) &= \sum_{\tau=0}^{t} V(\tau+1) - V(\tau) \\ &= \sum_{\tau=0}^{t} \mathbf{y}_0^T(\tau+1)\mathbf{P}\mathbf{y}_0(\tau+1) - \mathbf{y}_0^T(\tau)\mathbf{P}\mathbf{y}_0(\tau) \end{aligned} \tag{23}$$

Substituting the expression of $\mathbf{y}_0(\tau+1)$ into (23) and summing it up with (22), we obtain

$$\begin{aligned} J(t) + V(t) - V(0) = \sum_{\tau=0}^{t} \big(&\mathbf{y}_0^T(\tau)(\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T - \mathbf{P} + \mathbf{I})\mathbf{y}_0(\tau) \\ &+ \mathbf{v}^T(\tau)(-\gamma^2\mathbf{I} + \mathbf{L}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L})\mathbf{v}(\tau) \\ &- \mathbf{y}_0^T(\tau)\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L}\mathbf{v}(\tau) - \mathbf{v}^T(\tau)\mathbf{L}^T\ \mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\mathbf{y}_0(\tau)\big) \end{aligned} \tag{24}$$

This can be written in a quadratic form,

$$\begin{aligned} &J(t) + V(t) - V(0) \\ &= \sum_{\tau=0}^{t} \left[\mathbf{y}_0^T(\tau), \mathbf{v}^T(\tau)\right] \times \begin{bmatrix} \mathbf{I} + \mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T - \mathbf{P} & -\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L} \\ -\mathbf{L}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T & -\gamma^2\mathbf{I} + \mathbf{L}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{y}_0(\tau) \\ \mathbf{v}(\tau) \end{bmatrix} \end{aligned} \tag{25}$$

Because

$$\begin{bmatrix} \mathbf{I} + \mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T - \mathbf{P} & -\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L} \\ -\mathbf{L}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T & -\gamma^2\mathbf{I} + \mathbf{L}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L} \end{bmatrix} < 0$$

we have

$$\begin{aligned} J(t) < -V(t) + V(0) \Rightarrow J(t) < V(0) \\ \Rightarrow \sup_{t>0} \sum_{\tau=0}^{t} \mathbf{z}^T(\tau)\mathbf{z}(\tau) - \gamma^2\mathbf{v}^T(\tau)\mathbf{v}(\tau) < V(0) \end{aligned} \tag{26}$$

Choosing the constant in expression (10) not less than $V(0)$ leads to the conclusion. This concludes the proof of Lemma 1. □

Lemma 1 holds for non-symmetric $\mathbf{W}$, which corresponds to a directed graph. It gives a sufficient condition to meet the performance index (10). The condition is represented by a matrix inequality but not a linear one (recall that $\mathbf{L} = \mathbf{I} - \mathbf{W}$). Therefore, (20) is hard to solve directly. We next relax it to a LMI, for which powerful mathematic tools exist and can be used to find solutions [27]. This result is stated in the following theorem.

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Theorem 1

For a strongly connected, balanced and directed graph with a fixed topology, system (8) has a bounded consensus error and meets the performance index (10) if there exists a symmetric positive definite matrix $\mathbf{U} \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfying the following LMI:

$$\begin{bmatrix} -\mathbf{U} & \mathbf{S} & -\mathbf{T}_0\mathbf{L} & \mathbf{0} \\ \mathbf{S}^T & -\mathbf{U} & \mathbf{0} & \mathbf{U}\mathbf{T}_0 \\ -\mathbf{L}^T\mathbf{T}_0^T & \mathbf{0} & -\gamma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_0^T\mathbf{U} & \mathbf{0} & -\mathbf{I} \end{bmatrix} < 0 \tag{27}$$

where $\mathbf{S} = \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\mathbf{U}$.

Proof

The proof that system (8) has a bounded consensus error follows the argument in Remark 6. Now, we prove the rest part of Theorem 1. Because $\mathrm{diag}([\mathbf{I}, \mathbf{U}^{-1}, \mathbf{I}, \mathbf{I}])$ has full rank, we have

$$\begin{aligned} & \begin{bmatrix} -\mathbf{U} & \mathbf{S} & -\mathbf{T}_0\mathbf{L} & \mathbf{0} \\ \mathbf{S}^T & -\mathbf{U} & \mathbf{0} & \mathbf{U}\mathbf{T}_0 \\ -\mathbf{L}^T\mathbf{T}_0^T & \mathbf{0} & -\gamma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_0^T\mathbf{U} & \mathbf{0} & -\mathbf{I} \end{bmatrix} < 0 \\ \Leftrightarrow \quad & \mathrm{diag}([\mathbf{I}\,\mathbf{U}^{-1}\,\mathbf{I}\,\mathbf{I}]) \begin{bmatrix} -\mathbf{U} & \mathbf{S} & -\mathbf{T}_0\mathbf{L} & \mathbf{0} \\ \mathbf{S}^T & -\mathbf{U} & \mathbf{0} & \mathbf{U}\mathbf{T}_0 \\ -\mathbf{L}^T\mathbf{T}_0^T & \mathbf{0} & -\gamma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_0^T\mathbf{U} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \mathrm{diag}([\mathbf{I}\,\mathbf{U}^{-1}\,\mathbf{I}\,\mathbf{I}]) < 0 \\ \Leftrightarrow \quad & \left[\begin{array}{ccc|c} -\mathbf{U} & \mathbf{S}\mathbf{U}^{-1} & -\mathbf{T}_0\mathbf{L} & \mathbf{0} \\ \mathbf{U}^{-1}\mathbf{S}^T & -\mathbf{U}^{-1} & \mathbf{0} & \mathbf{T}_0 \\ -\mathbf{L}^T\mathbf{T}_0^T & \mathbf{0} & -\gamma^2\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{T}_0^T & \mathbf{0} & -\mathbf{I} \end{array}\right] < 0 \end{aligned} \tag{28}$$

Using the Schur complement, we obtain the following equivalent inequality:

$$\begin{aligned} & \begin{bmatrix} -\mathbf{U} & \mathbf{S}\mathbf{U}^{-1} & -\mathbf{T}_0\mathbf{L} \\ \mathbf{U}^{-1}\mathbf{S}^T & -\mathbf{U}^{-1} & \mathbf{0} \\ -\mathbf{L}^T\mathbf{T}_0^T & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{T}_0 \\ \mathbf{0} \end{bmatrix} (-\mathbf{I})^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{T}_0^T & \mathbf{0} \end{bmatrix} < 0 \\ \Leftrightarrow & \left[\begin{array}{c|cc} -\mathbf{U} & \mathbf{S}\mathbf{U}^{-1} & -\mathbf{T}_0\mathbf{L} \\ \hline \mathbf{U}^{-1}\mathbf{S}^T & -\mathbf{U}^{-1} + \mathbf{T}_0\mathbf{T}_0^T & \mathbf{0} \\ -\mathbf{L}^T\mathbf{T}_0^T & \mathbf{0} & -\gamma^2\mathbf{I} \end{array}\right] < 0 \end{aligned} \tag{29}$$

Using the Schur complement to this inequality, we equivalently obtain

$$\begin{bmatrix} -\mathbf{U}^{-1} + \mathbf{T}_0\mathbf{T}_0^T & \mathbf{0} \\ \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{U}^{-1}\mathbf{S}^T \\ -\mathbf{L}^T\mathbf{T}_0^T \end{bmatrix} (-\mathbf{U})^{-1} \begin{bmatrix} \mathbf{S}\mathbf{U}^{-1} & -\mathbf{T}_0\mathbf{L} \end{bmatrix} < 0 \tag{30}$$

Recalling $\mathbf{T}_0\mathbf{T}_0^T = \mathbf{I}$, $\mathbf{S} = \mathbf{T}_0\mathbf{W}\,\mathbf{T}_0^T\mathbf{U}$, defining $\mathbf{P} = \mathbf{U}^{-1}$ and then substituting them into the expression, we can finally obtain the expression of (27), which concludes the proof. □

Remark 7

Solving the LMI in (27) needs the global information of the graph Laplacian matrix. This step is carried out offline, as done in most of work using optimization methods [8, 20, 22]. Once the weighting matrix is obtained, the consensus protocol (4) can be used online to achieve consensus in the presence of quantization errors.

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For the probabilistic quantization scheme, further results can be drawn by exploiting the statistic properties of the probabilistic quantization error. On the motivation to consider the probabilistic quantization scheme, we have the following remark.

Remark 8
The probabilistic quantization scheme is widely used in existing work [9, 15] to help reduce the quantization effect. As pointed out in [28], the probabilistic quantization is essentially a form of dithered quantization method, which renders the quantization noise independent of the quantized data and reduces artifacts created by deterministic quantization. This motivates us to explore particularly the probabilistic quantization for the protocol (4).

For the probabilistic quantization scheme, we are more interested in the statistical performance of the protocol. We next study the steady state value of $E(\mathbf{z}(t)\mathbf{z}^T(t))$, which is the covariance matrix of $\mathbf{x}(t)$.

As argued in [15], the quantization error introduced by the probabilistic quantization scheme can be modeled as a random variable with zero mean and finite variance. The quantization errors are independent across sensors and across different iterations. Let σ^2 denote the quantization error variance of each node, we can express the covariance matrix of $\mathbf{v}(t)$ as $\sigma^2\mathbf{I}$. The deviation from consensus $\mathbf{z}(t)$, under the probabilistic quantization scheme, is also a random variable. We have the following theorem about $E(\mathbf{z}(t)\mathbf{z}^T(t))$.

Theorem 2
For system (8) on a strongly connected, balanced and directed graph with a fixed topology under the probabilistic quantization, the covariance matrix $E(\mathbf{z}(t)\mathbf{z}^T(t))$ converges to a constant matrix $\mathbf{\Sigma}_z$, whose vectorized form is given by

$$\mathrm{vec}(\mathbf{\Sigma}_z) = \left(\mathbf{T}_0^T \otimes \mathbf{T}_0^T\right)\left(\mathbf{I} - \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^{-1} \cdot ((\mathbf{T}_0\mathbf{L}) \otimes (\mathbf{T}_0\mathbf{L}))\mathrm{vec}(\mathbf{\Sigma}_v) \tag{31}$$

where $\mathbf{\Sigma}_v$ is the covariance matrix of $\mathbf{v}(t)$.

Proof
According to (11), we have

$$\begin{aligned}\mathbf{y}_0(t+1)\mathbf{y}_0^T(t+1) = {} & \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\mathbf{y}_0(t)\mathbf{y}_0^T(t)\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T + \mathbf{T}_0\mathbf{L}\mathbf{v}(t)\mathbf{v}^T(t)\mathbf{L}^T\mathbf{T}_0^T \\ & - \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\mathbf{y}_0(t)\mathbf{v}^T(t)\mathbf{L}^T\mathbf{T}_0^T - \mathbf{T}_0\mathbf{L}\mathbf{v}(t)\mathbf{y}_0^T(t)\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T\end{aligned} \tag{32}$$

Calculating the expected value on both sides, we obtain

$$E\left(\mathbf{y}_0(t+1)\mathbf{y}_0^T(t+1)\right) = \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T E\left(\mathbf{y}_0(t)\mathbf{y}_0^T(t)\right)\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T + \mathbf{T}_0\mathbf{L}E(\mathbf{v}(t)\mathbf{v}^T(t))\mathbf{L}^T\mathbf{T}_0^T \tag{33}$$

Defining a new variable $\mathbf{\Sigma}_{y0}(t) = E\left(\mathbf{y}_0(t)\mathbf{y}_0^T(t)\right)$ and noting that $E(\mathbf{v}(t)\mathbf{v}^T(t)) = \mathbf{\Sigma}_v$, which is a constant matrix, we have,

$$\mathbf{\Sigma}_{y0}(t+1) = \mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\mathbf{\Sigma}_{y0}(t)\mathbf{T}_0\mathbf{W}^T\mathbf{T}_0^T + \mathbf{T}_0\mathbf{L}\mathbf{\Sigma}_v\mathbf{L}^T\mathbf{T}_0^T \tag{34}$$

Use the Kronecker product to write $\mathbf{\Sigma}_{y0}(t+1)$ in a vector form,

$$\mathrm{vec}(\mathbf{\Sigma}_{y0}(t+1)) = \left(\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)\mathrm{vec}(\mathbf{\Sigma}_{y0}(t)) + ((\mathbf{T}_0\mathbf{L}) \otimes (\mathbf{T}_0\mathbf{L}))\mathrm{vec}(\mathbf{\Sigma}_v) \tag{35}$$

This is a linear time invariant system and therefore we can study its system matrix for the convergence. The eigenvalue of $\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)$ is $\lambda_i\lambda_j$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, where λ_i is the i-th eigenvalue of $\mathbf{T}_0\mathbf{W}\,\mathbf{T}_0^T$. In addition, $|\lambda_i| < 1$ as argued in Remark 6. Therefore, $\left|\mathrm{eig}\left(\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)\right| < 1$. Hence, $\mathbf{\Sigma}_{y0}(t)$ converges eventually. Denoting $\mathbf{\Sigma}_{y0}$ as

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its steady state value, we can calculate it easily by solving the steady state linear equation. After calculation, we have,

$$\mathrm{vec}(\mathbf{\Sigma}_{y0}) = \left(\mathbf{I} - \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^{-1}((\mathbf{T}_0\mathbf{L}) \otimes (\mathbf{T}_0\mathbf{L}))\mathrm{vec}(\mathbf{\Sigma}_v) \tag{36}$$

Together with $\mathbf{\Sigma}_z = \mathbf{T}_0^T\mathbf{\Sigma}_{y0}\mathbf{T}_0$, we obtain,

$$\begin{aligned}\mathrm{vec}(\mathbf{\Sigma}_z) &= \left(\mathbf{T}_0^T \otimes \mathbf{T}_0^T\right)\mathrm{vec}(\mathbf{\Sigma}_{y0})\\ &= \left(\mathbf{T}_0^T \otimes \mathbf{T}_0^T\right)\left(\mathbf{I} - \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^{-1}((\mathbf{T}_0\mathbf{L}) \otimes (\mathbf{T}_0\mathbf{L}))\mathrm{vec}(\mathbf{\Sigma}_v)\end{aligned} \tag{37}$$

This concludes the proof. □

Theorem 3

For system (8) on a strongly connected, balanced and directed graph with a fixed topology under the probabilistic quantization, if the performance index (10) is satisfied, the mean square deviation $E(\mathbf{z}^T(t)\mathbf{z}(t)/n)$ converges and is eventually upper bounded by $\gamma^2\sigma^2$. That is,

$$\lim_{t\to\infty} E\left(\frac{\mathbf{z}^T(t)\mathbf{z}(t)}{n}\right) < \gamma^2\sigma^2 \tag{38}$$

where $E(\mathbf{z}^T(t)\mathbf{z}(t)/n)$ is the expected value of $\mathbf{z}^T(t)\mathbf{z}(t)/n$, n is the number of nodes in the graph, σ^2 is the quantization error variance of each node and γ is defined in the performance index (10).

Proof

Because $E(\mathbf{z}(t)\mathbf{z}^T(t))$ converges (Theorem 2), we conclude $\lim_{t\to\infty} E\left(\frac{\mathbf{z}^T(t)\mathbf{z}(t)}{n}\right)$ exists by noting that $\mathbf{z}^T(t)\mathbf{z}(t) = \mathrm{trace}(\mathbf{z}(t)\mathbf{z}^T(t))$. According to the performance index (10), we have for all $t > 0$,

$$\sum_{\tau=0}^{t} \mathbf{z}^T(\tau)\mathbf{z}(\tau) - \gamma^2\mathbf{v}^T(\tau)\mathbf{v}(\tau) < \text{constant} \tag{39}$$

Calculating expected value on both sides and rewriting the expression, we obtain,

$$\sum_{\tau=0}^{t} E(\mathbf{z}^T(\tau)\mathbf{z}(\tau)) < t\gamma^2\mathrm{trace}(E(\mathbf{v}(\tau)\mathbf{v}^T(\tau))) + \text{constant} \tag{40}$$

For the probabilistic quantization, the covariance matrix of $\mathbf{v}$ is $\sigma^2\mathbf{I}$. Thus, we have $\mathrm{trace}(E(\mathbf{v}(\tau)\mathbf{v}^T(\tau))) = n\sigma^2$. Substituting this expression into inequality (40), dividing both side of the inequality with t and then calculating the limit as t goes to infinity, we have,

$$\lim_{t\to\infty} E\left(\frac{\mathbf{z}^T(t)\mathbf{z}(t)}{n}\right) < \gamma^2\sigma^2 + \lim_{t\to\infty}\frac{\text{constant}}{t} \tag{41}$$

Because $\lim_{t\to\infty}\frac{\text{constant}}{t} = 0$, we obtain the inequality (38). This concludes the proof. □

Remark 9

We can consider $\mathbf{z}(t)$ as the error from the average consensus. Then, $\sqrt{E\left(\frac{\mathbf{z}^T(t)\mathbf{z}(t)}{n}\right)}$ is the standard deviation of this consensus error. This theorem reveals that the error sensitivity is less than γ in the sense of standard deviation. In practice, we may want the steady-state mean square deviation $\lim_{t\to\infty} E\left(\frac{\mathbf{z}^T(t)\mathbf{z}(t)}{n}\right)$ upper bounded (say the desired upper bound is α_{ss}). According to (38), we can choose $\gamma = \frac{\sqrt{\alpha_{ss}}}{\sigma}$ in the design.

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5.2.1. Special case: undirected graph. For a strongly connected undirected graph, the weighting matrix is symmetric, and the steady-state standard deviation of the consensus error, $\lim_{t\to\infty} E(\mathbf{z}^T(t)\mathbf{z}(t))$, can be explicitly expressed in terms of the spectrum of the graph.

For system (8) on a strongly connected undirected graph, which is a special case of balanced and directed graphs, we have the following results according to Theorem 2,

$$\begin{aligned}
&\lim_{t\to\infty} E(\mathbf{z}^T(t)\mathbf{z}(t)) \\
&= \lim_{t\to\infty} \mathrm{trace}(E(\mathbf{z}(t)\mathbf{z}^T(t))) \\
&= \mathrm{trace}(\boldsymbol{\Sigma}_z) \\
&= \mathrm{vec}^T(\mathbf{I})\mathrm{vec}(\boldsymbol{\Sigma}_z) \\
&= \mathrm{vec}^T(\mathbf{I})\left(\mathbf{T}_0^T \otimes \mathbf{T}_0^T\right)\left(\mathbf{I} - \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^{-1} \cdot ((\mathbf{T}_0\mathbf{L}) \otimes (\mathbf{T}_0\mathbf{L}))\mathrm{vec}(\boldsymbol{\Sigma}_v)
\end{aligned} \tag{42}$$

With the Kronecker product property that $(\mathbf{B}^T \otimes \mathbf{A})\mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{AXB})$ for matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ of appropriate sizes, we have $\mathrm{vec}^T(\mathbf{I})\left(\mathbf{T}_0^T \otimes \mathbf{T}_0^T\right) = \mathrm{vec}^T\left(\mathbf{T}_0\mathbf{T}_0^T\right) = \mathrm{vec}^T(\mathbf{I})$ and $((\mathbf{T}_0\mathbf{L}) \otimes (\mathbf{T}_0\mathbf{L}))\mathrm{vec}(\boldsymbol{\Sigma}_v) = \mathrm{vec}\left(\mathbf{T}_0\mathbf{L}\boldsymbol{\Sigma}_v\mathbf{L}^T\mathbf{T}_0^T\right)$. Also noticing that $\boldsymbol{\Sigma}_v = \sigma^2\mathbf{I}$ for the identically independent quantization with the variance σ^2 for each node, we have,

$$\lim_{t\to\infty} E(\mathbf{z}^T(t)\mathbf{z}(t)) = \sigma^2\mathrm{vec}^T(\mathbf{I})\left(\mathbf{I} - \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^{-1}\mathrm{vec}\left(\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T\right) \tag{43}$$

The matrix $\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)$ is symmetric for undirected graph with $\mathbf{W} = \mathbf{W}^T$ and its eigenvalues locate strictly in the unit circle as pointed out in the proof of Theorem 2. Thus, we can express $\left(\mathbf{I} - \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^{-1}$ in the following geometric series,

$$\begin{aligned}
\left(\mathbf{I} - \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^{-1} &= \sum_{j=0}^{\infty}\left(\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right) \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)\right)^j \\
&= \sum_{j=0}^{\infty}\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j
\end{aligned} \tag{44}$$

Therefore,

$$\begin{aligned}
\lim_{t\to\infty} E(\mathbf{z}^T(t)\mathbf{z}(t)) &= \sigma^2\mathrm{vec}^T(\mathbf{I})\left(\sum_{j=0}^{\infty}\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\right)\mathrm{vec}\left(\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T\right) \\
&= \sigma^2\sum_{j=0}^{\infty}\left(\mathrm{vec}^T(\mathbf{I})\left(\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j \otimes \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\right)\mathrm{vec}\left(\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T\right)\right) \\
&= \sigma^2\sum_{j=0}^{\infty}\left(\mathrm{vec}^T(\mathbf{I})\mathrm{vec}\left(\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\right)\right) \\
&= \sigma^2\sum_{j=0}^{\infty}\mathrm{trace}\left(\mathbf{I}\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T \cdot \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\right) \\
&= \sigma^2\sum_{j=0}^{\infty}\mathrm{trace}\left(\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T \cdot \left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\right)
\end{aligned} \tag{45}$$

By exploiting the property that $\mathbf{T}_0\mathbf{1} = \mathbf{0}$, $\mathbf{T}_0\mathbf{T}_0^T = \mathbf{I}$ and $\mathbf{T}_0^T\mathbf{T}_0 = \mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{n}$, we obtain $\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j = \mathbf{T}_0\left(\mathbf{W}^j - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\mathbf{T}_0^T$ and further $\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j = \mathbf{T}_0\left(\mathbf{W}^j - \mathbf{W}^{j+1}\right)^2\mathbf{T}_0^T$

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after some trivial calculations. Notice that

$$\mathbf{T}\left(\mathbf{W}^{j}-\mathbf{W}^{j+1}\right)^{2}\mathbf{T}^{T}=\begin{bmatrix}\mathbf{T}_0\left(\mathbf{W}^{j}-\mathbf{W}^{j+1}\right)^{2}\mathbf{T}_0^{T} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{46}$$

where $\mathbf{T}$ is defined in (13). As $\mathbf{T}\mathbf{T}^T = \mathbf{I}$, (46) implies the $(n-1)\times(n-1)$ matrix $\mathbf{T}_0\left(\mathbf{W}^{j}-\mathbf{W}^{j+1}\right)^{2}\mathbf{T}_0^{T}$ has the same spectrum as the $n \times n$ matrix $\left(\mathbf{W}^{j}-\mathbf{W}^{j+1}\right)^{2}$ except the zero eigenvalue, that is, matrix $\mathbf{T}_0\left(\mathbf{W}^{j}-\mathbf{W}^{j+1}\right)^{2}\mathbf{T}_0^{T}$ has eigenvalues at $\left(\lambda_i^j-\lambda_i^{j+1}\right)^2$ for $i = 2, 3, \ldots, n$ with λ_i denoting the ith eigenvalue of $\mathbf{W}$. Therefore,

$$\begin{aligned}\operatorname{trace}\left(\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\mathbf{T}_0\mathbf{L}\mathbf{L}^T\mathbf{T}_0^T\cdot\left(\mathbf{T}_0\mathbf{W}\mathbf{T}_0^T\right)^j\right) &= \sum_{i=2}^{n}\left(\lambda_i^j-\lambda_i^{j+1}\right)^2\\ &= \sum_{i=2}^{n}\lambda_i^{2j}(1-\lambda_i)^2\end{aligned} \tag{47}$$

Thus,

$$\begin{aligned}\lim_{t\to\infty}E(\mathbf{z}^T(t)\mathbf{z}(t)) &= \sigma^2\sum_{j=0}^{\infty}\sum_{i=2}^{n}\lambda_i^{2j}(1-\lambda_i)^2\\ &= \sigma^2\sum_{i=2}^{n}\left((1-\lambda_i)^2\left(\sum_{j=0}^{\infty}\lambda_i^{2j}\right)\right)\\ &= \sigma^2\sum_{i=2}^{n}\left((1-\lambda_i)^2\frac{1}{1-\lambda_i^2}\right)\\ &= \sigma^2\sum_{i=2}^{n}\frac{1-\lambda_i}{1+\lambda_i}\end{aligned} \tag{48}$$

Consequently, we conclude that the inequality (38), for strongly connected undirected graph, is guaranteed by the following condition,

$$\frac{1}{n}\sum_{i=2}^{n}\frac{1-\lambda_i}{1+\lambda_i}<\gamma^2 \tag{49}$$

where λ_i denotes the ith eigenvalue of $\mathbf{W}$.

For an undirected graph with identically weighted edges, there is only one degree of freedom for the weighting matrix and it is of the form $\mathbf{W} = \mathbf{I} - k\mathbf{L}_0$ where $\mathbf{L}_0$ is the Laplacian matrix of the graph with unit edge weights. The weighting matrix design is summarized by the following theorem.

Theorem 4

For system (8) on a strongly connected undirected graph with a fixed topology under the probabilistic quantization, the mean square deviation $E(\mathbf{z}^T(t)\mathbf{z}(t)/n)$ is eventually upper bounded by $\gamma^2\sigma^2$ under the following condition,

$$0<k<\frac{2}{\mu_n}\min\left\{\frac{n\gamma^2}{n\gamma^2+n-1},\ 1\right\} \tag{50}$$

where $\mathbf{W} = \mathbf{I} - k\mathbf{L}_0$, $\mathbf{L}_0$ is the Laplacian matrix of the graph with unit edge weights and μ_n is its largest eigenvalue.

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Proof
The condition $k < \frac{2}{\mu_n}$ in (50) guarantees non-one eigenvalues of $\mathbf{W}$ locate strictly inside the unit circle and system (8) will not diverge under this condition. Now, to conclude the result, we show $k < \frac{2n\gamma^2}{\mu_n(n\gamma^2+n-1)}$ in inequality (50) is sufficient for (49).

$$\begin{aligned}\sum_{i=2}^{n}\frac{1-\lambda_i}{1+\lambda_i} < n\gamma^2 &\Leftarrow \sum_{i=2}^{n}\left(-1+\frac{2}{1+\lambda_i}\right) < n\gamma^2 \\ &\Leftarrow (n-1)(-1+\frac{2}{2-k\mu_n}) < n\gamma^2 \\ &\Leftarrow k < \frac{2n\gamma^2}{\mu_n(n\gamma^2+n-1)} \end{aligned} \tag{51}$$

Note that $\lambda_i \geqslant 1-k\mu_n$ for $i = 2, 3, \ldots, n$ is used in the derivation of the second step of (51). (51) completes the proof. □

Remark 10
According to the Perron–Frobenius Theorem [29], μ_n is upper bounded by $2\mathrm{deg}_{\max}$ with $\mathrm{deg}_{\max}$ denoting the maximum degree of $\mathbf{L}_0$. As a result, (50) holds for $0 < k < \frac{1}{\mathrm{deg}_{\max}} \min\left\{\frac{n\gamma^2}{n\gamma^2+n-1}, 1\right\}$.

Remark 11
The largest eigenvalue μ_n can be analytically obtained for some special topologies [30], such as complete graph, cycle topology, star topology and so on. In these cases, the upper bound of k in (50) can be explicitly obtained, as summarized in Table I.

5.3. *Directed graph with switching topology*

In some practices, failure of previous communication links and emergence of new communication connections may happen. Both will lead to a change of the network topology. Base on this consideration, we study directed graphs with switching topologies in this section. In this situation, we model the system as a switched system [31], where the switch is triggered by a switching signal $s(t)$.

Table I. The range of k for special topologies with the performance index γ. The weighting matrix is chosen to be $\mathbf{W} = \mathbf{I}-k\mathbf{L}_0$ with $\mathbf{L}_0$ denoting the Laplacian matrix of the corresponding graph with unit edge weights.

Topology	k
Complete graph	$0 < k < \frac{2}{n} \min\left\{\frac{n\gamma^2}{n\gamma^2+n-1}, 1\right\}$
Strongly regular graph $srg(n,d,a,b)$	$0 < k < \frac{4}{2n-(a-b)+\sqrt{(a-b)^2+4(d-b)}} \min\left\{\frac{n\gamma^2}{n\gamma^2+n-1}, 1\right\}$
Cycle	$0 < k < \frac{1}{2} \min\left\{\frac{n\gamma^2}{n\gamma^2+n-1}, 1\right\}$
Star	$0 < k < \frac{2}{n} \min\left\{\frac{n\gamma^2}{n\gamma^2+n-1}, 1\right\}$
Path	$0 < k < \frac{1}{2} \min\left\{\frac{n\gamma^2}{n\gamma^2+n-1}, 1\right\}$
Complete bipartite graph	$0 < k < \frac{2}{n} \min\left\{\frac{n\gamma^2}{n\gamma^2+n-1}, 1\right\}$

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In order to distinguish the difference of weighting matrix $\mathbf{W}$ associated with different topologies, in this section we use $G_{s(t)}$, $\mathbf{W}_{s(t)}$ and $\mathbf{L}_{s(t)}$ to represent the graph, weighting matrix and Laplacian matrix at time t. For a directed graph with a switching topology, we have similar results to Lemma 1 and Theorem 1.

Lemma 2
For a strongly connected, balanced and directed graph with a switching topology $G_{s(t)}$, system (8) satisfies the objective function (10) if there exists a common symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfying

$$\begin{bmatrix} \mathbf{\Gamma}_{11} & \mathbf{\Gamma}_{12} \\ \mathbf{\Gamma}_{21} & \mathbf{\Gamma}_{22} \end{bmatrix} < 0 \tag{52}$$

where

$$\mathbf{\Gamma}_{11} = \mathbf{I} + \mathbf{T}_0\mathbf{W}_{s(t)}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}_{s(t)}\mathbf{T}_0^T - \mathbf{P} \tag{53}$$

$$\mathbf{\Gamma}_{12} = -\mathbf{T}_0\mathbf{W}_{s(t)}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L}_{s(t)} \tag{54}$$

$$\mathbf{\Gamma}_{21} = -\mathbf{L}_{s(t)}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{W}_{s(t)}\mathbf{T}_0^T \tag{55}$$

$$\mathbf{\Gamma}_{22} = -\gamma^2\mathbf{I} + \mathbf{L}_{s(t)}^T\mathbf{T}_0^T\mathbf{P}\mathbf{T}_0\mathbf{L}_{s(t)} \tag{56}$$

and $s(t)$ denotes the switching signal.

Proof
The proof of this lemma is similar to Lemma 1. The difference is that both $\mathbf{W}_{s(t)}$ and $\mathbf{L}_{s(t)}$ share a common Lyapunov function $V(t) = \mathbf{y}_0^T(t)\mathbf{P}\mathbf{y}_0(t)$. □

Theorem 5
For a strongly connected, balanced and directed graph with a switching topology $G_{s(t)}$, system (8) has a bounded consensus error and achieves the performance index (10) if there exists a common symmetric positive definite matrix $\mathbf{U} \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfying the following LMI:

$$\begin{bmatrix} -\mathbf{U} & \mathbf{S}_{s(t)} & -\mathbf{T}_0\mathbf{L}_{s(t)} & \mathbf{0} \\ \mathbf{S}_{s(t)}^T & -\mathbf{U} & \mathbf{0} & \mathbf{U}\mathbf{T}_0 \\ -\mathbf{L}_{s(t)}^T\mathbf{T}_0^T & \mathbf{0} & -\gamma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_0^T\mathbf{U} & \mathbf{0} & -\mathbf{I} \end{bmatrix} < 0 \tag{57}$$

where $\mathbf{S}_{s(t)} = \mathbf{T}_0\mathbf{W}_{s(t)}\mathbf{T}_0^T\mathbf{U}$ and $s(t)$ denotes the switching signal.

Proof
This theorem follows from Lemma 2 and can be similarly proved as Theorem 1. □

For graphs with fixed topologies, we studied the steady state covariance matrix $E(\mathbf{z}(t)\mathbf{z}^T(t))$ of the deviation in Theorem 3. However, the steady state value may not exist for the case on a switching graph due to the topology changes. Therefore, no conclusions on the steady state value is made for the switching case. Nevertheless, Theorem 5 guarantees average consensus with an H_∞ performance for the switching topology.

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6. SIMULATIONS

In this section, we give three simulation examples, one on a directed fixed graph, one on a directed switching graph and the third one on an undirected fixed graph. As shown in [9, 15], the probabilistic quantization, compared with the deterministic quantization, has the advantage to reduce the consensus disagreement in a statistical sense. In order to demonstrate that the proposed method is able to further reduce the consensus disagreement, we use the probabilistic quantization scheme in the simulation. Nevertheless, the performance index is also guaranteed for the deterministic quantization according to Theorem 1 and Theorem 5. We compare our proposed method with the maximum-degree weights [14], the least-mean-square consensus (LMSC) method [8] and the modified LMSC method in the simulation. It is worth noting that although all theorems are presented for balanced directed graphs, the design applies to general (unbalanced) directed graphs as shown in the following examples, which is carried out by imposing a constraint at the beginning of the design as described in Remark 1.

6.1. *Example 1 (directed, fixed graph)*

In this example, we study the case of a directed fixed graph and show the design procedures using the proposed method. There are 10 nodes in a directed network. The information flow of this network is shown in Figure 1. In the simulation, we set $\eta = 2$ and $b = 2$ for the probabilistic quantizer. As proved in [28], the variance σ^2 of the quantization error in this situation satisfies $\sigma^2 \leqslant \frac{\delta^2}{4}$ with $\delta = \frac{2\eta}{2^b - 1}$. After calculations, we obtain $\sigma \leqslant 0.67$. The design objective is to keep the steady-state mean square deviation from the average consensus (i.e., $\lim_{t\to\infty} E\left(\frac{\mathbf{z}^T(t)\mathbf{z}(t)}{n}\right)$ in inequality (38)) no larger than 0.6. We choose $\gamma = 1.1$ to be the L_2 gain between the consensus deviation and the quantization error. The LMI is constructed based on Theorem 1 and numerically solved by MATLAB. The matrix $\mathbf{T}$ in the LMI (27) is chosen according to Remark 5. That is, the matrix $\mathbf{T}$ is selected so that its rows equal to the normalized eigenvectors of the matrix $\mathbf{1}\mathbf{1}^T$ with the last row equal to the eigenvector $\frac{\mathbf{1}^T}{\sqrt{n}}$. Initially, the matrix $\mathbf{U}$ in the LMI (27) is chosen as the identity matrix. However, with $\gamma = 1.1$, no solution is returned by solving the LMI (27), which indicates that the initial choice of $\mathbf{U}$ as an identity matrix is not a good trial. We then relax γ to a relative larger value 2 and choose $\mathbf{U}$ as the identity matrix to solve the LMI (27). In this case, a solution of the weighting matrix $\mathbf{W}$ can be found. Based on the obtained weighting matrix $\mathbf{W}$, we treat the LMI (27) as an inequality with an unknown variable $\mathbf{U}$ and solve it to reach a new $\mathbf{U}$. Then we use the new $\mathbf{U}$ to solve the LMI to find $\mathbf{W}$. This operation is repeated several times to reduce the value of γ iteratively. After five iterations,

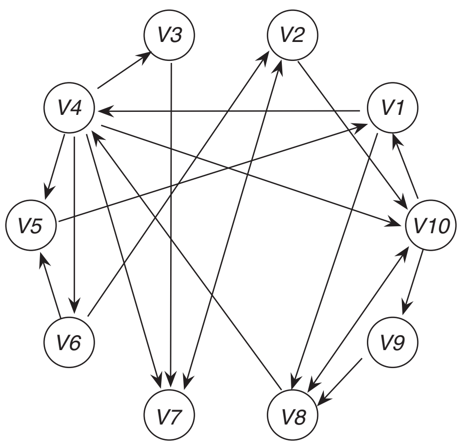

Figure 1. Information flow of network in the example 1. Circles labeled from $V1$ to $V10$ represent the nodes in the network. The arrow on the line indicates the direction of communication.

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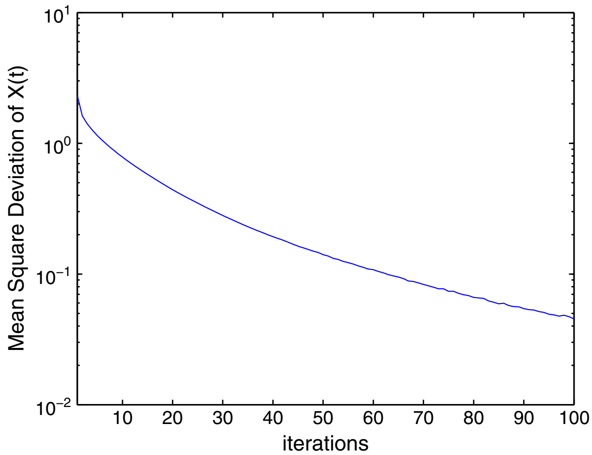

Figure 2. Mean square deviations of $\mathbf{x}(t)$ in log scales versus iteration numbers.

we obtain the weighting matrix $\mathbf{W}$ with the given index γ. Note that we use this numerical procedure to find a possible solution of the LMI for a given γ. Minimizing the parameter γ in LMI is out of the scope of the current paper, for which existing methods can be found in [27]. The obtained $\mathbf{U}$ (9×9 square matrix) and $\mathbf{W}$ (10×10 square matrix) are shown in the box. The position of nonzero entries in the resulting weighting matrix $\mathbf{W}$ is consistent with the directed topology of the graph shown in Figure 1. We measure the performance using $\mathbf{z}^T(t)\mathbf{z}(t)/n$. Results are averaged over 500 Monte Carlo runs with independently generated initial states. Results are shown in Figure 2. From this figure we can see that $\mathbf{z}^T(t)\mathbf{z}(t)/n$ reduces gradually. Finally, the value is lower than our predefined upper bound 0.6.

6.2. *Example 2 (directed, switching graph)*

The second example studies the directed switching graph case. There are four nodes in a network with four different topologies switching in order in every 20 iterations. The information flow of the four different topologies is shown in Figure 3. Each node uses the consensus protocol (4) to update the state value. Given $\gamma = 1.6$, we design the weighting matrix for each topology. We compare the result with the case using maximum-degree weights. The initial values of each node are generated according to the Gaussian distribution with zero mean and unit variance. We measure the performance by $\mathbf{z}^T(t)\mathbf{z}(t)/n$. Results are averaged over 500 Monte Carlo runs with independently generated initial states. In the simulation, we set $\eta = 2$ and $b = 2$ for the quantizer. Figure 4 shows the empirical mean square deviation of $\mathbf{x}(t)$ versus the number of iterations for the case with maximum-degree weights and that designed with our method. From Figure 4, we can see the mean square deviation of our method that is much lower than the case using maximum-degree weights.

6.3. *Example 3 (undirected, fixed graph)*

As mentioned earlier, most studies on average consensus with quantized communication assumes that the topology is undirected and fixed. To facilitate comparison, we consider here a wireless sensor network with an undirected and fixed topology. The network is generated by using a random geometric graph model [2, 14], in which nodes are uniformly distributed in a two-dimensional unit area and nodes with distance less than $\sqrt{\ln n/n}$ (n is the number of sensor nodes) are thought to be neighbors. In the simulation, we choose $n = 25$. The constructed graph in this simulation is shown in Figure 5. The quantization parameters are chosen as $\eta = 2$, $b = 4$, respectively. In each run, the state value of each node is initialized according to a Gaussian distribution with zero mean and unit variance and results are averaged over 500 Monte Carlo runs with independently generated initial states. We use $\gamma = 1$ to design the weighting matrix $\mathbf{W}$, which is a 25×25 symmetric stochastic matrix. We compare our method with the maximum-degree weights method used in [14],

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$$\mathbf{U} = \begin{bmatrix} 0.1246 & 0.0103 & 0.0031 & 0.0060 & 0.0034 & 0.0028 & -0.0023 & -0.0102 & -0.0711 \\ 0.0103 & 0.4109 & 0.0121 & -0.0100 & 0.0185 & 0.0118 & 0.0136 & -0.0828 & 0.2692 \\ 0.0031 & 0.0121 & 0.0521 & 0.0293 & 0.0319 & 0.0338 & 0.1296 & 0.0252 & -0.0133 \\ 0.0060 & -0.0100 & 0.0293 & 0.1137 & 0.0126 & 0.0261 & 0.1315 & 0.0016 & 0.0272 \\ 0.0034 & 0.0185 & 0.0319 & 0.0126 & 0.0741 & 0.0247 & 0.1288 & -0.0095 & -0.0197 \\ 0.0028 & 0.0118 & 0.0338 & 0.0261 & 0.0247 & 0.0459 & 0.1176 & 0.0213 & -0.0126 \\ -0.0023 & 0.0136 & 0.1296 & 0.1315 & 0.1288 & 0.1176 & 0.5807 & 0.1124 & 0.0074 \\ -0.0102 & -0.0828 & 0.0252 & 0.0016 & -0.0095 & 0.0213 & 0.1124 & 0.1647 & -0.0010 \\ -0.0711 & 0.2692 & -0.0133 & 0.0272 & -0.0197 & -0.0126 & 0.0074 & -0.0010 & 0.3777 \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} 0.8539 & 0 & 0 & 0.0546 & 0 & 0 & 0 & 0.0915 & 0 & 0 \\ 0 & 0.6012 & 0 & 0 & 0 & 0 & 0.3602 & 0 & 0 & 0.0386 \\ 0 & 0 & 0.9879 & 0 & 0 & 0 & 0.0121 & 0 & 0 & 0 \\ 0 & 0 & 0.0121 & 0.8996 & 0.0349 & 0.0133 & 0.0185 & 0 & 0 & 0.0216 \\ 0.0403 & 0 & 0 & 0 & 0.9597 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0080 & 0 & 0 & 0.0054 & 0.9867 & 0 & 0 & 0 & 0 \\ 0 & 0.3908 & 0 & 0 & 0 & 0 & 0.6092 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0458 & 0 & 0 & 0 & 0.6067 & 0 & 0.3475 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0832 & 0.9168 & 0 \\ 0.1058 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2186 & 0.0832 & 0.5924 \end{bmatrix}$$

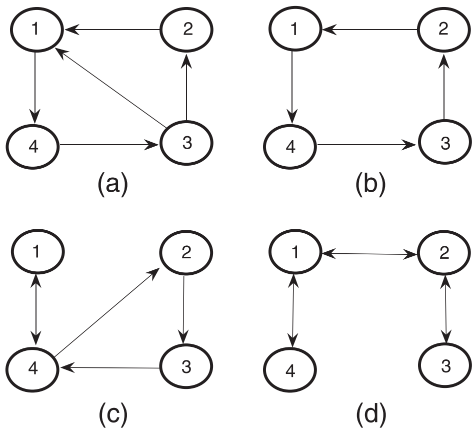

Figure 3. Four switching topologies in the example. The topology starts from graph (a) at time $t = 0$ and switches to the next one in every 20 iterations in the order of (a)→(b)→(c)→(d)→(a)...

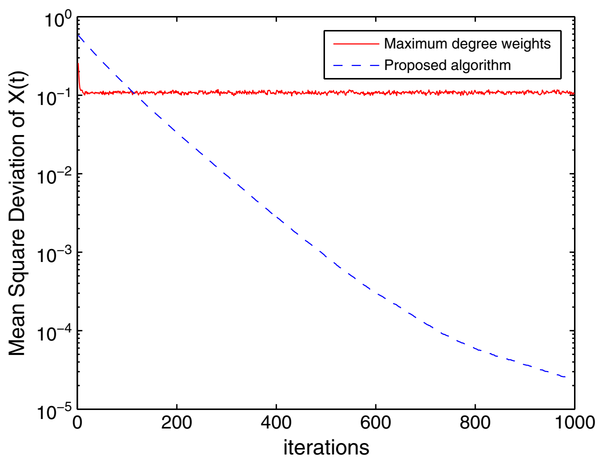

Figure 4. Mean square deviations of $\mathbf{x}(t)$ in log scales versus iteration numbers.

the so-called LMSC method proposed in [8] and the so-called modified LMSC method with different parameters and use the mean square deviations from the initial average as the measurement. As to the LMSC method, the weighting matrix is obtained by gradient descending search given by (19) in [8]. In Figure 6, we can see a clear advantage of using the weighting matrix obtained by the proposed algorithm over the LMSC method and the maximum-degree weights method. After

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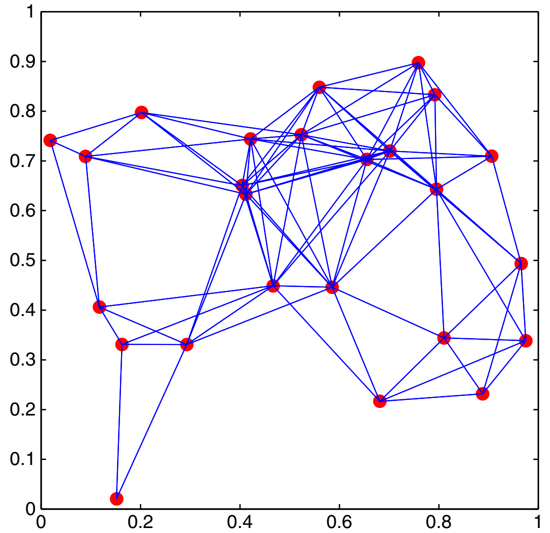

Figure 5. The random geometric graph used in the simulation. The number of nodes is $n = 25$. Communication radius is $\sqrt{\ln n/n}$.

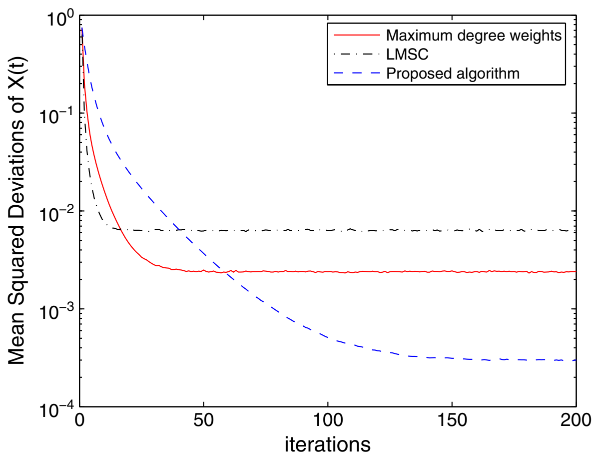

Figure 6. Mean square deviations of $\mathbf{x}(t)$ from the initial average in log scales versus iteration numbers.

adequate iterations, the disagreement from consensus of the proposed algorithm is less than them. Note that LMSC method optimizes the weighting matrix based on the least square deviation criterion under topological constraints. However, because of the additive noise of the LMSC method does not properly model the quantization error, the performance of the LMSC method is not as good as our method in the simulation.

6.4. *Comparison with other optimization methods*

In [8], the LMSC method is formulated as the following consensus problem with additive noises:

$$\mathbf{x}(t+1) = \mathbf{W}\mathbf{x}(t) + \omega(t) \tag{58}$$

where $\omega(t)$ is an independent and identically-distributed additive noise with zero mean and constant variance. For the convenience of comparison, the consensus problem with a probabilistic quantization can be written as the following by substituting $\mathbf{L} = \mathbf{I} - \mathbf{W}$ into (7):

$$\mathbf{x}(t+1) = \mathbf{W}\mathbf{x}(t) - \mathbf{v}(t) + \mathbf{W}\mathbf{v}(t) \tag{59}$$

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where $\mathbf{v}(t)$ is an independent and identically-distributed random variable with zero mean and finite variance. Because the quantization error is a white noise, the LMSC method proposed in [8] can be extended to handle quantized communications. We compare the performances of our method with the extended LMSC method. Because we do not see any publication about the application of LMSC method in the quantized communication case, we show the detail of the extension in the following.

Above all, the modified LMSC method employs the following assumptions:

1. The graph is undirected. That is, the weighting matrix satisfies that $\mathbf{W} = \mathbf{W}^T$.
2. The topology is fixed.

Note that the assumption of undirected graph is needed to guarantee the convexity of the problem. The assumption of fixed topology is required to ensure the existence of the steady-state mean square deviation.

The problem is formulated as follows:

$$\begin{array}{ll} \min & \delta_{ss}(\mathbf{W}) \\ \text{s.t.} & \mathbf{W} = \mathbf{W}^T, \mathbf{W}\mathbf{1} = \mathbf{1}, \mathbf{W} \in \mathbb{S}, \left\| \mathbf{W} - \frac{\mathbf{1}\mathbf{1}^T}{n} \right\| < 1 \end{array} \tag{60}$$

where n is the number of nodes in the graph, and δ_{ss} is the steady-state mean square deviation. $\left\| \mathbf{W} - \frac{\mathbf{1}\mathbf{1}^T}{n} \right\| < 1$ is the constraint for convergence and $\mathbf{W} \in \mathbb{S}$ is the topology constraint. Note that $\delta_{ss}(\mathbf{W})$ has the following expression:

$$\delta_{ss} = \frac{1}{n}\text{trace}\left(\mathbf{\Sigma}_{ss}\right) \tag{61}$$

where $\mathbf{\Sigma}_{ss} = \lim_{t\to\infty} \mathbf{\Sigma}_z(t)$ denotes the steady-state variance of $\mathbf{z}(t)$ and $\mathbf{\Sigma}_z(t)$ denotes the variance of $\mathbf{z}(t)$ in (8). That is, $\mathbf{\Sigma}_z(t) = E(\mathbf{z}(t)\mathbf{z}^T(t))$. As proved in (48), the objective function in (60) has the following form:

$$\delta_{ss}(\mathbf{W}) = \frac{\sigma^2}{n} \sum_{i=2}^{n} \frac{1-\lambda_i}{1+\lambda_i} \tag{62}$$

where λ_i denotes the ith largest eigenvalue of $\mathbf{W}$ and σ^2 is the quantization error variance of each node. With the expression (62), we are able to prove the convexity of the problem (60), by using the theory of convex spectral functions [32]. To solve this optimization problem, we can write it as the following convex optimization problem after some trivial operations:

$$\begin{array}{ll} \min & g(\mathbf{W}) = \frac{2}{n}\text{trace}\left(\mathbf{I} + \mathbf{W} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)^{-1} \\ \text{s.t.} & -\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{n} + \mathbf{W} < 0 \\ & -\mathbf{I} + \frac{\mathbf{1}\mathbf{1}^T}{n} - \mathbf{W} < 0 \\ & \mathbf{W} = \mathbf{I} - \sum_{k=1}^{m} \omega_k \mathbf{a}_k \mathbf{a}_k^T \end{array} \tag{63}$$

where $\mathbf{a}_k$ denotes the kth column of the incidence matrix of the graph, ω_k denotes the weight of the kth edge and m denotes the total amount of edges.

The modified LMSC method solves problem (63) numerically to find the optimal weighting matrix. In implementation, as often carried out in semi-definite programming, we use the interior point method to relax problem (63) into an unconstrained optimization problem with the following objective function:

$$\min f(\mathbf{W}) = \frac{2}{n}\text{trace}\left(\mathbf{I} + \mathbf{W} - \frac{\mathbf{1}\mathbf{1}^T}{n}!\right)^{-1} - \mu_0 \ln\left(\det\left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{n} + \mathbf{W}\right)\right) - \mu_0 \ln\left(\det\left(\mathbf{I} + \frac{\mathbf{1}\mathbf{1}^T}{n} - \mathbf{W}\right)\right) \tag{64}$$

where μ_0 is a positive constant and $\det(\cdot)$ denotes the determinant of a square matrix. By searching in the negative gradient direction, we can find the optimal weighting matrix to problem (64), which

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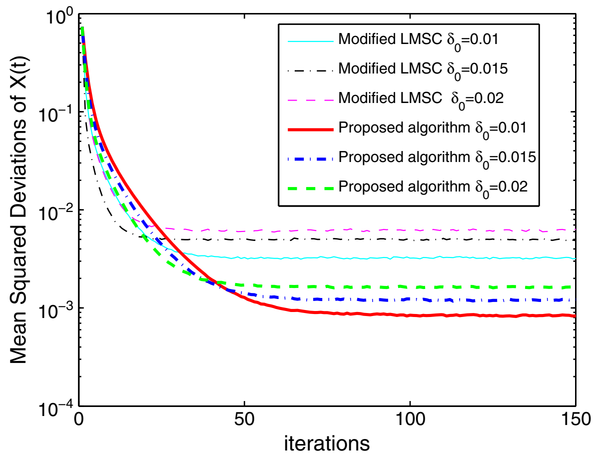

Figure 7. Comparisons of the proposed method and the modified LMSC method. x-label represents the number of iterations while y-label represent the mean square deviations of $\mathbf{x}(t)$ in log scales.

is a proper approximation, by choosing a small enough μ_0, to the original problem (60). As observed from (62), a weighting matrix with eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$ very close to but smaller than 1, always satisfies the constraints in (63) and can have the objective function arbitrarily close to zero. In this way, the optimal solution of $\mathbf{W}$ will be infinitely close to the identity matrix. However, such a $\mathbf{W}$ reduces the information exchange between neighbor nodes and needs infinitely long time for convergence. In other words, the ideal solution obtained by the modified LMSC method is infeasible.

To make the modified LMSC method feasible, we set up an *allowed steady-state mean square deviation*, δ_0, as the design criterion and run the modified LMSC method until this criterion is satisfied. In the following, we use both our method and the modified LMSC method to design weighting matrices for $\delta_0 = 0.01$, $\delta_0 = 0.015$ and $\delta_0 = 0.02$, respectively and compare the results. As shown in Figure 7, the mean square deviation of our method is lower than that of the modified LMSC method when the algorithm converges. It should be stressed that the modified LMSC method can only deal with undirected fixed graphs. In contrast, by defining the performance index from the L_2 gain perspective, the case with switching topology and the case on directed graph can be included into our framework as stated in Theorem 1 and Theorem 5 and as depicted in the simulation Example 1 and Example 2.

7. CONCLUSION

In this paper, average consensus on a directed graph with quantized communication was studied. We examined this problem for both fixed and switching topologies. A robust weighting matrix design was proposed to reduce the effect of the quantization error. Theoretical analysis led to a set of LMIs which can be used to solve the weighting matrices. For fixed topology with probabilistic quantization, the mean square deviation was proved to converge and its upper bound was derived. Simulation results were given to verify the effectiveness of the method.

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