2014 11th IEEE International
Conference on Control & Automation (ICCA)
June 18-20, 2014. Taichung, Taiwan

Discrete-time Consensus Filters on Directed Switching Graphs

Shuai Li and Yi Guo

Abstract— **We consider discrete-time distributed estimation on a directed graph with switching topologies. Motivated by a recent PI consensus filter, we modify the protocol and remove the constraint on bi-directional information exchange and gain conditions that require global topological information. The protocol is then extended to switching topologies. Convergence results for time-invariant inputs under both balanced directed and general directed graphs are given for switching topologies. Satisfactory simulation results are shown to validate theoretical claims.**

Index Terms— **Distributed estimation, consensus filter, directed graph, switching topology.**

I. INTRODUCTION

Distributed estimation is a fundamental problem in networked systems. Direct applications of conventional estimation methods often need all-to-all communications, which causes large communication burdens. Much attention has been paid recently to consensus or gossip algorithms to relax the all-to-all communication requirements to neighbor-to-neighbor communications. Although the consensus protocol [1] and the gossip algorithm [2] are able to estimate global quantities in a distributed way, they are in lack of explicit input signals thus cannot track time-varying inputs without re-initialization. However, applications of distributed Kalman filter [3] and cooperative control [4] need continuous estimation of global quantities and call for distributed estimation schemes for time-varying signals. Consensus filters open a promising door to such dynamic distributed estimation [4]–[6].

In general cases of distributed sensing, each agent has a different input and the goal is to track the average of the set of inputs. The design of consensus filters aims to find dynamic evolutions converging to the input average in a distributed way. In [3], [5], [7], Olfati-saber and co-authors introduced a distributed low-pass consensus filter and a distributed high-pass consensus filter, which are able to track the average of inputs of all sensors in a network. In the case that the input to sensors are not identical, estimation error exists. Progresses were made in [4], [6], [8] to reduce the estimation error. In [4], [6], Freeman *et. al.* proposed a PI consensus filter in continuous-time form, which is able to converge accurately to the average of the inputs when they are time-invariant. In [9], the discrete-time counterpart of the PI consensus filter was derived and successfully applied to dynamically merging feature-based maps in robot networks. However, this discrete-time PI consensus filter has a gain condition requiring information on the spectrum of the Laplacian matrix of the graph, which is a centralized quantity and normally unavailable in a distributed network. In [10], a consensus filter is proposed for distributed map merging under switching topologies. However, the time interval between switching is larger than the convergence time of the filter, *i.e.*, switching happens when the filter almost converges to the desired value. As to arbitrary switching, the system may lose stability even though it is convergent and stable under a fixed topology [11].

In this paper, we introduce new consensus filters for distributed averaging on a network and extend our previous works on continuous-time consensus filters [12], [13] to discrete-time versions. By a proper design of the co-state dynamics to compensate the network switching, accurate convergence results are reached on directed graphs with arbitrary switching topologies. Compared to existing work, the contribution of the paper includes:

1) The proposed protocols work on general directed graphs, which extends the undirected graph limitation in [9].

2) The proposed consensus filters allow arbitrary switching topologies with guaranteed convergence, which is in contrast to existing consensus filter work in this field [3]–[8].

3) The gain condition for the proposed filters is very mild and requires no global information on the graph structure, while the gain condition proposed in the recent work [9] depends on the graph Laplacian.

The rest of this paper is organized as follows: In Section II, preliminaries on graph theory and consensus filters are introduced. A consensus filter on a general graph with a fixed topology is proposed and theoretically analyzed in Section III. In Section IV, the second consensus filter is proposed and theoretically analyzed to solve distributed estimation problem on switching topologies. Numerical simulations presented in Section V validate the theoretical results. Section VI concludes this paper.

II. PRELIMINARIES

A. Fundamental Knowledge on Graphs

A directed graph $G(V, E, A)$ is denoted by (V, E, A), where V is the set of nodes, E is the set of edges with $E \subseteq V \times V$, and $A = [a_{ij}]$ is the weighted adjacency matrix. The

This work was partially supported by the National Science Foundation under Grants IIS-1218155 and CNS-1318748.

The authors are with Department of Electrical and Computer Engineering, Stevens Institute of Technology, Hoboken, NJ 07030, USA. {lshuai,yi.guo}@stevens.edu

in-degree and out-degree of a node in the directed graph is defined as $\deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji}$ and $\deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}$ respectively. The directed graph G is said to be balanced if the in-degree equals the out-degree for each node in the graph. A special case of balanced graph is undirected graph, which bears the property of $a_{ji} = a_{ij}$ for all i, j. A directed graph G is called strongly connected if there always exists a sequence of consecutive edges starting from a given node i to another given node j, where node i and node j could be any node in the graph only if $i \neq j$. A directed graph G is called connected if there is an undirected path between any pair of nodes. The degree matrix $\Delta = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ij} = 0$ for all $i \neq j$ and $\Delta_{ii} = \mathrm{d}_{out}(v_i)$ for all i. The Laplacian matrix L of the graph G is defined as $L = \Delta - A$. For a directed graph, the rank of its Laplacian matrix is equal to $(n-1)$, where n is the dimension of the Laplacian matrix L if the graph contains a spanning tree. The stochastic matrix of the graph G is defined as $P = [p_{ij}]$ such that $p_{ij} = 0$ for $j \notin \mathbb{N}(i)$, $p_{ij} \geq 0$ for $\forall i, j$, and $\sum_{j \in \mathbb{N}(i)} p_{ij} = 1$ for $\forall i$. For a balanced graph, containing a spanning tree is equivalent to being connected. A directed graph with topology series $\{G_1(V_1, E_1, A_1), G_2(V_2, E_2, A_2), ...G_k(V_k, E_k, A_k)\}$ with $V_1 = V_2 = ... = V_k = V$, is called jointly-containing-spanning-tree if the union of the topologies $\sum_{i=1}^{k} G_i$, defined as $\sum_{i=1}^{k} G_i = G(V, \bigcup_{i=1}^{k} E_i, \sum_{i=1}^{k} A_k)$, has a spanning tree [14]. Particularly, if all topologies in the topology series are balanced, $\sum_{i=1}^{n} G_i$ having a spanning tree [14] is equivalent to being connected. In this case, this set of topologies is called jointly connected [15].

B. Graph Centrality

Centrality defines the relative importance of a node on a graph. There are several different measures of centrality, such as degree centrality, betweenness centrality, pagerank centrality [16], [17]. In this paper, we particularly consider the pagerank centrality. The pagerank centrality of node i has the following definition:

$$\alpha(i) d_{out}(v_i) = \sum_{j \in \mathbb{N}(i)} w_{ji} \alpha(j) \tag{1}$$

where $\alpha(i) \geq 0$ denotes the pagerank centrality of the ith node, $d_{out}(v_i)$ represents the out-degree of node i, w_{ji} is the weight for the edge from i to j. For all nodes in the graph, the vector $\alpha = [\alpha(1), \alpha(2), ..., \alpha(m)]^T$ satisfies,

$$\alpha^T L = 0 \tag{2}$$

As the Laplacian matrix of a graph always has a zero eigenvalue, Eq. (2) implies the pagerank centrality vector α is the left eigenvector of L corresponding to the zero eigenvalue (referred to as zero left eigenvector from now on). Clearly, the centralities of all nodes on a balanced connected graph are identical since $\mathbf{1} = [1, 1, ..., 1]^T$ is always the zero left eigenvector of L for balanced graphs. In addition, it can be justified that the centrality of a node with zero out-degree equals zero, meaning that this node has no impact to others since there is no out flow from it.

C. An Existing Consensus Filter

In [6], Freeman et al. proposed an average PI consensus filter in continuous time. This filter is extended to discrete time in [9] for undirected graph with a fixed topology, which writes,

$$\begin{aligned} x(n+1) &= (1 - h\gamma)x - hLx + hL\lambda(n) + h\gamma u \\ \lambda(n+1) &= -hLx(n) + \lambda(n) \end{aligned} \tag{3}$$

where $u \in \mathbb{R}^m$ is the input vector, $x(n) \in \mathbb{R}^m$ is the decision variable at time n and $\lambda(n) \in \mathbb{R}^m$ is the co-state, L is the Laplacian matrix of the communication graph, $h \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+$ are constants satisfying

$$\frac{3}{2}\mu_{max}(L) \leq \gamma \leq \frac{3}{2h} \tag{4}$$

where $\mu_{max}(L)$ is the maximum eigenvalue of the Laplacian matrix L.

Remark 1: The consensus filter (3) provides a means for distributed estimation of global quantities. Compared with consensus protocols, which does not have an explicit input, Protocol (3) can track the estimation under time-varying input signals. Also, compared with some existing consensus filters, such as the low-pass consensus filter [3]–[8], Protocol (3) converges to the average of static inputs ideally. However, there are still some limitations with Protocol (3):

1. This protocol limits to undirected communication graphs with a fixed topology. However, directed graph (includes undirected graph as a special case) and switching graphes are more general descriptions of distributed networks with directional communication, possible link failure and re-connections.

2. The gain condition in (4) depends on the maximum eigenvalue of the Laplacian matrix L, which is a global information and is difficult to obtain.

III. Consensus Filtering on Direct Graphs with a Fixed Topology

A. The Protocol

We proposed the following discrete-time consensus filter for direct graphs with a fixed topology,

$$\begin{aligned} x_i(n+1) &= (1-\gamma)u_i(n) + \gamma \sum_{j \in \mathbb{N}(i) \cup \{i\}} w_{ij} x_j(n) \\ &- (1-\gamma) \sum_{k=0}^{n} \sum_{j \in \mathbb{N}(i)} w_{ij} \big(x_i(k) - x_j(k)\big) \end{aligned} \tag{5}$$

where $x_i(n)$ represents the ith decision variable at time n and is initialized randomly, $u_i(n)$ is the ith input, $\mathbb{N}(i)$ represents the neighboring set of the ith node, $-1 < \gamma < 1$, $w_{ij} > 0$

for $j \in \mathbb{N}(i)$, $w_{ii} = 1 - \sum_{j \in \mathbb{N}(i)} w_{ij} > 0$, and $w_{ij} = 0$ for $j \notin \mathbb{N}(i) \cup \{i\}$.

Its realization writes,

$$\begin{aligned} x_i(n+1) &= (1-\gamma)u_i(n) + \gamma \sum_{j \in \mathbb{N}(i) \cup \{i\}} w_{ij} x_j(n) \\ &\quad -(1-\gamma)\lambda_i(n+1) \\ \lambda_i(n+1) &= \lambda_i(n) + \sum_{j \in \mathbb{N}(i)} w_{ij}\big(x_i(n) - x_j(n)\big) \end{aligned} \quad (6)$$

with the initialization $\lambda_i(0) = 0$ for all i.

Protocol (5) can be written in the following matrix form,

$$x(n+1) = (1-\gamma)u(n) + \gamma P x(n) - (1-\gamma)\sum_{k=0}^{n} L x(k) \quad (7)$$

where $P = [w_{ij}]$ is the stochastic matrix defined on the graph and L is a Laplacian matrix defined as $L = I - P$ with I denoting the identity matrix of an appropriate size.

B. Convergence Results

We have the following theorem for this protocol (7) with time-invariant inputs.

Theorem 1: For the consensus protocol (7) or its realization (6), where L is a Laplacian matrix defined as $L = I - P$ with P being a stochastic matrix on the communication graph, $-1 < \gamma < 1$ and α is the pagerank centrality of the graph (i.e. α is a zero left eigenvector of L), $x(n)$ converges to average consensus with the common value $\frac{\alpha^T u}{\alpha^T \mathbf{1}}$ for time-invariant $u(n) = u$ (where u is a constant vector).

Proof: See Appendix I. ■

Remark 2: For balanced graphs, the pagerank centrality is identical in all dimensions thus the protocol (7) reaches distributed averaging with identical weights.

IV. Consensus Filtering on a Graph with Switching Topologies

In this section, we modify the proposed consensus filter by introducing an additional term to compensate for the topological switching. We first consider the case on a balanced graph and then investigate the result on a general graph.

A. The Protocol

For graphs with switching topologies, we propose the following protocol,

$$\begin{aligned} x_i(n+1) &= (1-\gamma)u_i(n) + \gamma x_i(n) \\ &\quad -(1-\gamma)\sum_{k=0}^{n} \sum_{j \in \mathbb{N}(i,k)} w_{ij}(k)\big(x_i(k) - x_j(k)\big) \\ &\quad -\gamma \sum_{k=0}^{n} \sum_{j \in \mathbb{N}(i,k)} w_{ij}(k)\big(x_i(k) - x_i(k-1) \\ &\quad -x_j(k) + x_j(k-1)\big) \end{aligned} \quad (8)$$

where $x_i(n)$ represents the ith decision variable at time n, $u_i(n)$ is the ith input, $\mathbb{N}(i,k)$ represents the neighboring set the ith node at time k, $-1 < \gamma < 1$, $w_{ij}(k) > 0$ for $j \in \mathbb{N}(i,k)$ and $w_{ij}(k) = 0$ otherwise.

Its realization writes,

$$\begin{aligned} x_i(n+1) &= (1-\gamma)u_i(n) + \gamma x_i(n) \\ &\quad -(1-\gamma)\lambda_i(n+1) - \gamma y_i(n+1) \\ \lambda_i(n+1) &= \lambda_i(n) + \sum_{j \in \mathbb{N}(i,n)} w_{ij}(n)\big(x_i(n) - x_j(n)\big) \\ y_i(n+1) &= y_i(n) + \sum_{j \in \mathbb{N}(i,n)} w_{ij}(n)\big(x_i(n) - x_i(n-1) \\ &\quad -x_j(n) + x_j(n-1)\big) \end{aligned} \quad (9)$$

with the initialization $\lambda_i(0) = 0$ and $y_i(0) = 0$ for all i.

In a compact matrix form, the protocol writes,

$$\begin{aligned} x(n+1) &= (1-\gamma)u(n) + \gamma x(n) - (1-\gamma)\sum_{k=0}^{n} L(k)x(k) \\ &\quad -\gamma \sum_{k=0}^{n} L(k)\big(x(k) - x(k-1)\big) \end{aligned} \quad (10)$$

where $-1 < \gamma < 1$, $L(k)$ is the Laplacian matrix at time k.

B. Convergence Results for A Balanced Graph with Switching Topology

On the proposed protocol (10) with time-invariant inputs, we have the following theorem,

Theorem 2: For the consensus protocol (10) or its realization (9), where $-1 < \gamma < 1$, $L(k)$ is a Laplacian matrix defined as $L(k) = I - P(k)$ with $P(k)$ being a stochastic matrix on the communication graph at time k, $x(n)$ converges to average consensus with the common value $\frac{\mathbf{1}^T u}{l}$ (where l is the dimension of the vector u) for time-invariant u, provided there exists an infinite sequence of uniformly bounded, non-overlapping time intervals, across which the graph is balanced and jointly connected.

Proof: See Appendix II. ■

C. Convergence Results for A General Directed Graph with Switching Topology

For a general graph with switching topologies, the switching family may not share a common pagerank centrality. In this situation, the discrete-time protocol (10) still reaches consensus. However, the common value is a result combining the impact of the topological structure across time. On this point, we have the following theorem.

Theorem 3: For the consensus protocol (10) or its realization (9), where $-1 < \gamma < 1$, $L(k)$ is a Laplacian matrix defined as $L(k) = I - P(k)$ with $P(k)$ being a stochastic matrix on the communication graph at time k, $x(n)$ converges to consensus for time-invariant $u(n) = u$ (u is a constant vector), provided that there exists an infinite

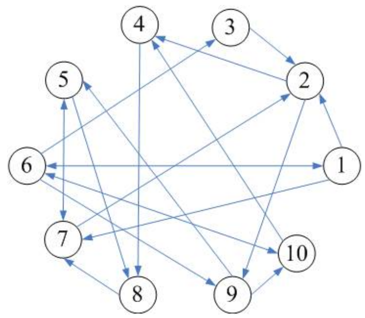

Fig. 1. A directed graph with a fixed topology

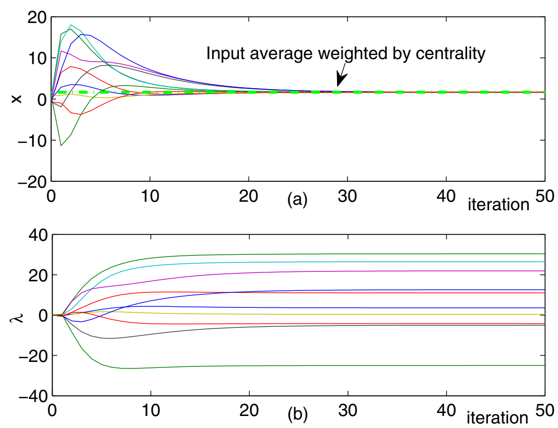

Fig. 2. Protocol (7) on a directed graph with a fixed topology with time-invariant inputs: (a). Time profile of $x(n)$. (b). Time profile of $\lambda(n)$.

sequence of uniformly bounded, non-overlapping time intervals, across which the graph is jointly-containing-spanning-tree. The common value of consensus is $\frac{\gamma\mu_\infty}{1-\gamma}$, where

$$\mu_\infty = \frac{1}{n}\mathbf{1}^T \lim_{n\to\infty} \prod P(n)P(n-1)...P(1)(\frac{x(1)}{\gamma} - x(0))$$

.

Proof: See Appendix III. ■

V. Simulations

In this section, we use simulations to validate the theoretical conclusions. We consider the proposed protocols under a fixed topology and switching topologies with time-invariant inputs.

For the case with fixed topology, we perform simulations on a small scale network with 10 nodes to show the performance (the network topology is shown in Fig.1). For simplicity, the stochastic matrix $P = [w_{ij}]$ is chosen such that $w_{ij} = \frac{1}{1+d(i)}$ for $j \in \mathbb{N}(i) \cup \{i\}$ with $d(i)$ denoting the number of inflow links connected to node i [18], [19]. The graph centrality in this case can be computed as $\alpha =$ $[0.5446,\ 0.0901,\ 0.0451,\ 0.0113,\ 0.0225,\ 0.7211,\ 0.0376,$ $0.0225,\ 0.1915,\ 0.3662]$. In the simulation, the input u is set as $u = [5.2312,\ 32.0100,\ 12.6290,\ 28.0824,\ 23.5652,$

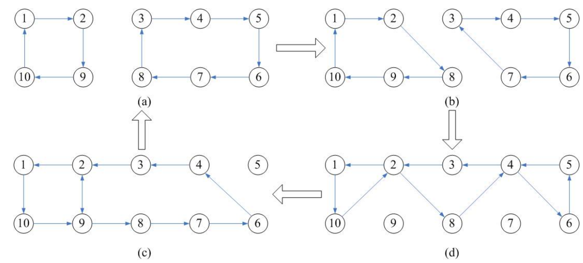

Fig. 3. The balanced graph with switching topologies used in the simulation study.

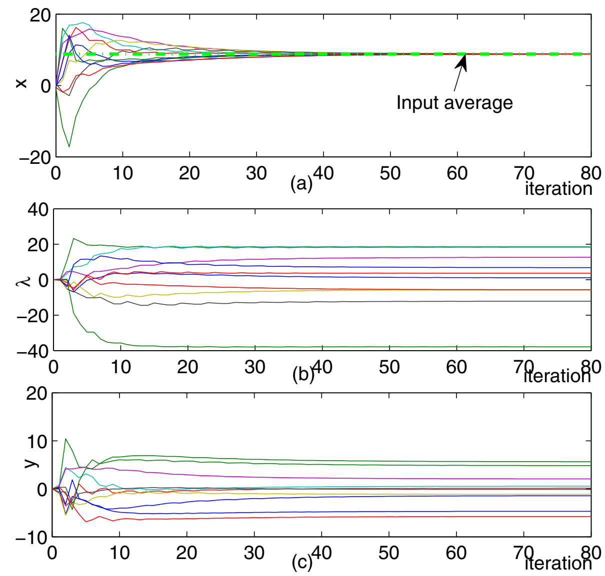

Fig. 4. Protocol (10) on a balanced graph with switching topologies with time-invariant inputs: (a). Time profile of $x(n)$. (b). Time profile of $\lambda(n)$. (c) Time profile of $y(n)$.

$2.0058,\ -3.4135,\ 14.2183,\ -23.3582,\ -2.5974]$. Simulation results are plotted in Fig. 2 by running protocol (7) with $\gamma = 0.5$. From this figure, it can be observed that x successfully reaches consensus with the common value at the input average weighted by centrality.

For the case with balanced switching topologies, the four different topologies indicated in Fig. 3 are employed with each topology running for one iteration in turn in a cycle. Each existing link is assigned with the weight 0.4. This simulation considers the same time-invariant input u as in the fixed topology case, and $\gamma = 0.5$. As shown in Fig. 4, x converges to the desired average by running Protocol (10) with time-invariant inputs.

For the general directed graph, we choose the switching topologies as shown in Fig. 5, which are not balanced. With the same input and the same γ as in the case with balanced switching topologies, as shown in Fig. 6 x in Protocol (10), also converges to consensus for time-invariant inputs. The time evolution of the co-states are also plotted in Fig. 6.

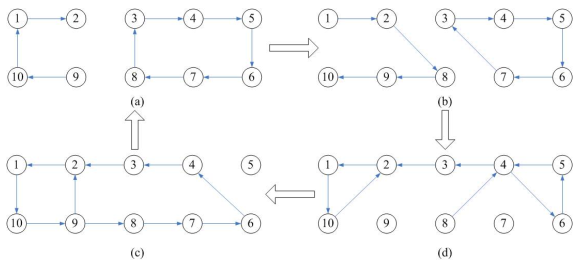

Fig. 5. The general graph graph with switching topologies used in the simulation study.

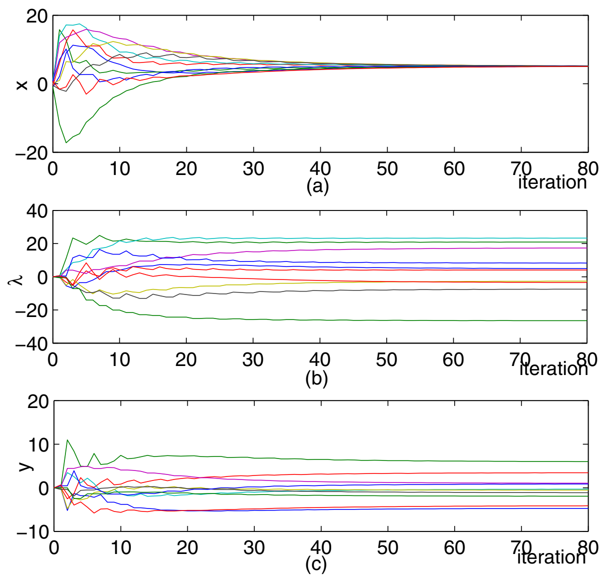

Fig. 6. Protocol (10) on a general graph with switching topologies with time-invariant inputs: (a). Time profile of $x(n)$. (b). Time profile of $\lambda(n)$. (c) Time profile of $y(n)$.

VI. Conclusion

In this paper, consensus filters running on switching directed graphs were investigated. Theoretical results proved the convergence to the weighted average of inputs in different scenarios. Numerical simulations validated the results.

Appendix I
Proof of Theorem 1

We first define $k_0 = \frac{1-\gamma}{\gamma}$, then $\gamma = \frac{1}{1+k_0}$. Substituting γ with k_0 in (7) yields,

$$x(n+1) - x(n) = -k_0\big(x(n+1) - u(n)\big) - Lx(n) - k_0 \sum_{k=0}^{n} Lx(k) \quad (11)$$

Define $s(n) = x(n) + k\sum_{k=0}^{n} x(k)$. Then we have

$$\begin{aligned} s(n+1) - s(n) &= x(n+1) - x(n) + k_0 x(n+1) \\ &= -k_0\big(x(n+1) - u(n)\big) - Lx(n) - k_0 \textstyle\sum_{k=0}^{n} Lx(k) \\ &\quad + k_0 x(n+1) = k_0 u(n) - L\big(x(n) + k_0 \textstyle\sum_{k=0}^{n} x(k)\big) \\ &= k_0 u(n) - Ls(n) \end{aligned} \quad (12)$$

Therefore,

$$s(n+1) = k_0 u(n) + (I - L)s(n) = k_0 u(n) + Ps(n) \quad (13)$$

For the time instant one step before, we have,

$$s(n) = k_0 u(n-1) + Ps(n-1) \quad (14)$$

Subtracting (14) from (13) yields the following by noting that $u(n) = u(n-1) = u$,

$$s(n+1) - s(n) = P\big(s(n) - s(n-1)\big) \quad (15)$$

That is, for the new variable $p(n) = s(n) - s(n-1)$, its dynamics follow the linear consensus protocol and converge to the average weighted by the zero left eigenvector of P, i.e, $\lim_{n\to\infty} p(n) = \frac{\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T p(0)$. Multiplying α^T on both sides of (12) yields,

$$\alpha^T p(n+1) = k_0 \alpha^T u(n) = k_0 \alpha^T u \quad (16)$$

Therefore, we conclude that $\lim_{n\to\infty} p(n) = \frac{\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T p(n) = \frac{k_0\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T u$. Now, let us consider $p(n) = s(n) - s(n-1) = x(n) + k_0\sum_{i=0}^{n} x(i) - x(n-1) - k_0\sum_{i=0}^{n-1} x(i) = x(n) - x(n-1) + k_0 x(n) = (1+k_0)x(n) - x(n-1)$. Thus, we have

$$x(n) = \frac{1}{1+k_0}x(n-1) + \frac{1}{1+k_0}p(n) \quad (17)$$

with $\lim_{n\to\infty} p(n) = \frac{k_0\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T u$. Noting that the linear system (17) with $|\frac{1}{1+k_0}| < 1$ (i.e., $|\gamma| < 1$) is BIBO and ultimately converges to $\lim_{n\to\infty} x(n) = \frac{\mathbf{1}}{k_0}\lim_{n\to\infty} p(n) = \frac{\mathbf{1}}{\alpha^T\mathbf{1}}\alpha^T u$, meaning that the protocol (7) reaches average consensus weighted by the zero left eigenvector of L.

Appendix II
Proof of Theorem 2

Define $k_0 = \frac{1-\gamma}{\gamma}$ and then $k_0 > 0$, $\gamma = \frac{1}{1+k_0}$. Eq. (10) can be written as the following by multiplying $(1+k_0)$ on both sides,

$$\begin{aligned} (1+k_0)x(n+1) &= x(n) + k_0 u(n) - k_0 \sum_{k=0}^{n} L(k)x(k) \\ &\quad - \sum_{k=0}^{n} L(k)\big(x(k) - x(k-1)\big) \end{aligned} \quad (18)$$

Define $s(n) = x(n) + k_0 \sum_{k=0}^{n} x(k)$ and $p(n) = s(n) - s(n-1)$. With (18), we have,

$$\begin{aligned} p(n) &= (1+k_0)x(n) - x(n-1) \\ &= k_0 u(n-1) - k_0 \textstyle\sum_{t=0}^{n-1} L(k)x(k) \\ &\quad - \textstyle\sum_{k=0}^{n-1} L(k)\big(x(k) - x(k-1)\big) \end{aligned} \tag{19}$$

Also, from Eq. (19), we have

$$\begin{aligned} p(n+1) = k_0 u(n) - k_0 \textstyle\sum_{t=0}^{n} L(k)x(k) - \textstyle\sum_{k=0}^{n} L(k)\big(x(k) \\ -x(k-1)\big) \end{aligned} \tag{20}$$

Accordingly, we have,

$$\begin{aligned} p(n+1) - p(n) = -k_0 L(n)x(n) - L(n)\big(x(n) \\ -x(n-1)\big) = -L(n)\big(k_0 x(n) + x(n) - x(n-1)\big) \\ = -L(n)p(n) \end{aligned} \tag{21}$$

Therefore,

$$p(n+1) = \big(I - L(n)\big)p(n) = P(n)p(n) \tag{22}$$

According to the results on discrete time consensus protocols, we concludes that $p(n)$ converges to consensus as n goes to infinity. Multiplying $\mathbf{1}^T$, which is the common zero left eigenvector of $L(k)$ at all time instants, on both sides of (20) yields that $\mathbf{1}^T p(n+1) = k_0 \mathbf{1}^T u(n) = k_0 \mathbf{1}^T u$ as the $\mathbf{1}^T L(k) = \mathbf{0}^T$ for all k_0 for balanced graphs. Therefore, we conclude that

$$\lim_{n\to\infty} p(n) = \frac{k_0 \mathbf{1}^T u}{l} \mathbf{1} \tag{23}$$

with l representing the dimension of the vector u. Recalling the definition $s(n) = x(n) + k_0 \sum_{k=0}^{n} x(k)$ and $p(n) = s(n) - s(n-1)$, we have $p(n) = x(n) - x(n-1) + k_0 x(n)$, i.e., $x(n) = \frac{1}{1+k_0} x(n-1) + \frac{1}{1+k_0} s(n) = \gamma x(n-1) + \gamma s(n)$. For this BIBO system with $s(n)$ as the input and $x(n)$ as the output, under the condition that $|\gamma| < 1$, we know that $x(n)$ is bounded and $x(n)$ stabilizes to $\lim_{n\to\infty} x(n) = \frac{\gamma}{1-\gamma} \frac{k_0 \mathbf{1}^T u}{l} \mathbf{1} = \frac{1}{k_0} \frac{k_0 \mathbf{1}^T u}{l} \mathbf{1} = \frac{\mathbf{1}^T u}{l} \mathbf{1}$, which is the average consensus of u. This concludes the result.

Appendix III
Proof of Theorem 3

Define k_0 such that $\gamma = \frac{1}{1+k_0}$. Then, define $s(n) = x(n) + k_0 \sum_{k=0}^{n} x(k)$, $p(n) = s(n) - s(n-1) = (1+k_0)x(n) - x(n-1)$. By following the same procedure as in the proof of Theorem 2, we get $p(n+1) = \big(I - L(n)\big)p(n) = P(n)p(n)$. For this linear difference equation, $\lim_{n\to\infty} p(n) = \lim_{n\to\infty} \prod P(n)P(n-1)...P(1)p(1) = \lim_{n\to\infty} \prod P(n) \cdot P(n-1)...P(1)\big(\frac{x(1)}{\gamma} - x(0)\big)$. Note that $p(n+1) = P(n)p(n)$ reaches consensus for $p(n)$ according to the conclusions on discrete-time linear consensus protocols. Therefore, $\lim_{n\to\infty} p(n) = \frac{\mathbf{1}}{n} \mathbf{1}^T \lim_{n\to\infty} \prod P(n)P(n-1)...P(1)\big(\frac{x(1)}{\gamma} - x(0)\big) = \mathbf{1}\mu_\infty$. Note that $x(n) = \gamma x(n-1) + \gamma p(n)$. This difference equation with $p(n)$ as the input converges provided that $-1 < \gamma < 1$. The ultimate value of $x(n)$ can be obtained as $\lim_{n\to\infty} x(n) = \frac{\gamma}{1-\gamma} \lim_{n\to\infty} p(n) = \frac{\gamma \mu_\infty}{1-\gamma} \mathbf{1}$. This concludes the result.

References

[1] L. Xiao, S. Boyd, and S.J. Kim. Distributed average consensus with least-mean-square deviation. *Journal of Parallel and Distributed Computing*, 67(1):33–46, 2007.

[2] A.G. Dimakis, S. Kar, J. Moura, M.G Rabbat, and A. Scaglione. Gossip algorithms for distributed signal processing. *Proceedings of the IEEE*, 98(11):1847–1864, 2010.

[3] R. Olfati-Saber. Distributed kalman filtering and sensor fusion in sensor networks. In P. Antsaklis and P. Tabuada, editors, *Networked Embedded Sensing and Control*, Lecture Notes in Control and Information Sciences, pages 157–167. Springer, 2006.

[4] P. Yang, R.A. Freeman, and K.M. Lynch. Multi-agent coordination by decentralized estimation and control. *IEEE Transactions on Automatic Control*, 53(11):2480–2496, 2008.

[5] R. Olfati-saber and J.S. Shamma. Consensus filters for sensor networks and distributed sensor fusion. In *44th IEEE Conference on Decision and Control*, pages 6699–6703, 2005.

[6] R.A. Freeman, P. Yang, and K.M. Lynch. Stability and convergence properties of dynamic average consensus estimators. In *45th IEEE Conference on Decision and Control*, pages 338 –343, 2006.

[7] R. Olfati-Saber. Distributed kalman filtering for sensor networks. In *46th IEEE Conference on Decision and Control*, pages 5492–5498, 2007.

[8] H. Bai, Randy A. Freeman, and Kevin M. Lynch. Robust dynamic average consensus of time-varying inputs. In *49th IEEE Conference on Decision and Control*, pages 3104–3109, 2010.

[9] R. Aragues, J. Cortes, and C. Sagues. Distributed consensus on robot networks for dynamically merging feature-based maps. *IEEE Transactions on Robotics*, 28(4):840–854, 2012.

[10] R. Aragues, J. Cortes, and C. Sagues. Distributed consensus algorithms for merging feature-based maps with limited communication. *Robotics and Autonomous Systems*, 59(3-4):163–180, 2011.

[11] H. Bai and M. Arcak. Instability mechanisms in cooperative control. *IEEE Transactions on Automatic Control*, 55(1):258 –263, 2010.

[12] S. Li and Y. Guo. Dynamic consensus estimation of weighted average on directed graphs. *International Journal of Systems Science*, 75:1–15, 2013.

[13] S. Li and Y. Guo. Distributed consensus filter on directed graphs with switching topologies. In *American Control Conference*, pages 6151–6156, 2013.

[14] T. Li and J. Zhang. Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises. *IEEE Transactions on Automatic Control*, 55(9):2043–2057, 2010.

[15] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[16] P. Bonacich. Some unique properties of eigenvector centrality. *Social Networks*, 29(4):555–564, 2007.

[17] M.E.J. Newman. Mathematics of networks. In S.N. Durlauf and L.E. Blume, editors, *The New Palgrave Dictionary of Economics*. Palgrave Macmillan, Basingstoke, 2008.

[18] T. Vicsek, A. Czirók, J.E. Ben, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, 1995.

[19] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.