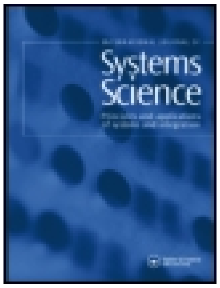

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Dynamic consensus estimation of weighted average on directed graphs

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Dynamic consensus estimation of weighted average on directed graphs

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Recent applications call for distributed weighted average estimation over sensor networks, where sensor measurement accuracy or environmental conditions need to be taken into consideration in the final consensused group decision. In this paper, we propose new dynamic consensus filter design to distributed estimate weighted average of sensors' inputs on directed graphs. Based on recent advances in the filed, we modify the existing proportional-integral consensus filter protocol to remove the requirement of bi-directional gain exchange between neighbouring sensors, so that the algorithm works for directed graphs where bi-directional communications are not possible. To compensate for the asymmetric structure of the system introduced by such a removal, sufficient gain conditions are obtained for the filter protocols to guarantee the convergence. It is rigorously proved that the proposed filter protocol converges to the weighted average of constant inputs asymptotically, and to the weighted average of time-varying inputs with a bounded error. Simulations verify the effectiveness of the proposed protocols.

Keywords: distributed estimation; consensus filter; directed graphs; weighted average

1. Introduction

Distributed estimation is a fundamental problem in networked systems, such as sensor networks (Akyildiz, Su, Sankarasubramaniam, and Cayirci 2002), mobile robot networks (Cao, Fukunaga, and Kahng 1997; Wieland, Kim, and Allgöwer 2011; Zhang, Wang, and Guo 2012), social networks (Scott 2000), and other process networked systems (Zhang, Shi, and Mehr 2011, 2012; Zhang, Shi, and Liu 2013). Direct applications of conventional estimation methods often need all-to-all communications, which causes large communication burdens. Much attention has been paid recently to consensus or gossip algorithms to relax the all-to-all communication requirements to neighbour-to-neighbour communications. In this paper, we present new distributed consensus filter algorithms for directed graphs.

Xiao, Boyd, and Lall (2005) directly applied average consensus protocol for the distributed sensor fusion to reach a final estimation with least mean square errors. This protocol does not have explicit input and cannot track the average of time-varying inputs, for which we refer to as 'static consensus estimation'. By treating the input signal as a virtual leader, Ren (2007) proposed a proportional derivative like controller to track a common time-varying signal distributively. This method requires an accurate knowledge of the derivative of the states of neighbours, which may not be realistic to implement in practice. To remove this limitation, Cao, Ren, and Li (2009) presented a discrete time protocol, which requires the state of one-step before from the neighbours. In more general cases of distributed sensing, each agent has a different input and the goal is to track the average of the set of inputs. Olfati-Saber and Shamma (2005) introduced a distributed low-pass consensus filter and a distributed high-pass consensus filter, which are able to track the average of inputs from all sensors in a network. In the case that the input of the sensors are not identical, estimation error exists even for a set of constant inputs. Olfati-Saber (2006, 2007) made progresses to reduce the estimation error. Freeman, Yang, and Lynch (2006) and Yang, Freeman, and Lynch (2008) proposed a proportional-integral (PI) consensus filter, which is able to converge accurately to the average of the inputs when the inputs are time-invariant. Examining the PI filter (Freeman et al. 2006; Yang et al. 2008) in frequency domain, the integral term introduces a zero zero, which cancels out the zero pole introduced by the constant input. This idea is generalised to ideally track the average of time-varying inputs by exploiting the internal model principle (Bai, Freeman, and Lynch 2010).

While the above-mentioned work study average consensus filter, every sensor's input is considered equal weight in the final consensused group estimation. In real-world applications, due to sensor measurement accuracy or other environmental conditions, one may want to filter sensors' inputs with some pre-defined or online adjusted weights. One recent work by our group (Zhang et al. 2011) shows that using desired weights for each sensor according to

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different channel conditions in communication can increase sensing efficiency in the application of distributed cooperative spectrum sensing for a radio network. However, there are few literature studying weighted consensus estimation. Particularly for directed graphs, introducing desired weights in the filtering protocol breaks the symmetric structure and brings imbalance of information flow, which makes the convergence analysis more challenging. While our earlier work (Zhang et al. 2011) proposed weighted average consensus protocol for constant inputs, we focus on *dynamic* estimation using consensus filter techniques for time-varying inputs in the current paper.

We study weighted consensus filter on directed graphs. We first define the problem as to distributively estimate the direct average or weighted average of input signals, which may be constant or time-varying, over a directed graph. For the case of direct average (i.e., average with identical weights of each sensor), based on recent advances on consensus filters, we modify the PI consensus filter (Freeman et al. 2006), and remove the requirement of bi-directional exchange of neighbouring gains. Sufficient gain conditions are obtained for the filter parameters to guarantee the convergence of the proposed protocol. We then extend it to the weighted average case, where similar gain conditions on filter parameters are given. It is rigorously proved that the proposed filter protocol converges to the weighted average of constant inputs asymptotically, and to the weighted average of time-varying inputs with a bounded error. Simulation results demonstrate the effectiveness of the method.

This work extends existing results in two ways: (1) the communication topology is extended to directed graphs comparing to existing work (Olfati-Saber and Shamma 2005), and the bi-directional exchange of neighbouring gains is removed comparing to the existing work by Freeman et al. (2006) and (2) average consensus is extended to weighted average consensus. The extension is not straightforward in comparison with existing average consensus filter design on undirected or balanced graphs, as the convergence of weighted average consensus filters on directed graphs does not hold unconditionally due to the imbalance of corresponding Laplacian matrices. In this aspect, sufficient gain conditions are obtained on filter parameters to guarantee the convergence of the proposed filter protocols.

The rest of the paper is organised as follows. In Section 2, we briefly review graph theory preliminaries and an existing PI average consensus filter protocol. Then in Section 3, we present motivating examples and define two problems to be solved in the paper. The main results are presented in Sections 4 and 5, where distributed averaging with identical weights and non-identical weights are discussed, respectively. Simulation results are provided in Section 6. Finally, the paper is concluded by brief remarks in Section 7.

2. Preliminaries

2.1. Graph theory

A directed graph $G(V, E, A)$ is denoted by (V, E, A), where V is the set of nodes, E is the set of edges with $E \subseteq V \times V$, and $A = [a_{ij}]$ is the weighted adjacency matrix. The in-degree and out-degree of a node in the directed graph is defined as $\deg_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ji}$ and $\deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ij}$, respectively. The directed graph G is said to be balanced if the in-degree equals to the out-degree for each node in the graph. A special case of balanced graphs is undirected graph, which bears the property of $a_{ji} = a_{ij}$ for all i, j. A directed graph G is called strongly connected if there always exist a sequence of consecutive edges starting from a given node i to another given node j, where node i and node j could be any node in the graph only if $i \neq j$. The degree matrix $\Delta = [\Delta_{ij}]$ is a diagonal matrix with $\Delta_{ij} = 0$ for all $i \neq j$ and $\Delta_{ii} = \deg_{\text{out}}(v_i)$ for all i. The Laplacian matrix L of the graph G is defined as $L = \Delta - A$. For a strongly connected graph, the rank of its Laplacian matrix is equal to $(n - 1)$ where n is the dimension of the Laplacian matrix L.

2.2. PI average consensus filter

Freeman et al. (2006) proposed an average consensus filter, which reads as follows:

$$\begin{aligned}
\dot{x}_i(t) &= -\gamma x_i(t) - \sum_{j \neq i} a_{ij}(t)(x_i(t) - x_j(t)) \\
&\quad + \sum_{j \neq i} b_{ji}(t)(\lambda_i(t) - \lambda_j(t)) + \gamma u_i(t), \\
\dot{\lambda}_i(t) &= -\sum_{j \neq i} b_{ij}(t)(x_i(t) - x_j(t)), \qquad (1)
\end{aligned}$$

where $u_i(t) \in \mathbb{R}$ is the input, $x_i(t) \in \mathbb{R}$ is the decision variable and $\lambda_i(t) \in \mathbb{R}$ is the co-state, $\gamma \in \mathbb{R}^+$ is a constant parameter, and $a_{ij}(t)$ and $b_{ij}(t)$ are piecewise continuous time-varying gains.

The compact matrix form of this protocol writes

$$\dot{x}(t) = -L_P(t)x(t) - \gamma(x(t) - u(t)) + L_I^T(t)\lambda(t), \qquad (2a)$$

$$\dot{\lambda}(t) = -L_I(t)x(t), \qquad (2b)$$

where $u(t) \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, and $\lambda(t) \in \mathbb{R}^n$ are the input vector, the decision variable vector, and the co-state variable vector, respectively. $L_P(t)$ and $L_I(t)$ are Laplacian matrices constructed by $[a_{ij}(t)]$ and $[b_{ij}(t)]$, respectively.

As proved by Freeman et al. (2006), the protocol (2) converges to a consensus value that is the average of the inputs under certain network conditions (characterised by mathematical conditions on matrices $L_I(t)$ and $L_P(t)$). Note that both $L_I(t)$ and $L_I^T(t)$ appear on the right side of (2) as coefficients (corresponding to the fact that both $b_{ij}(t)$ and

$b_{ji}(t)$ appear in (1)). This indicates that 'weight information must be communicated between agents in addition to the estimator state values' as claimed by Freeman et al. (2006). In this paper, we modify the above consensus filter and remove this requirement of bi-directional communication between agents, so that the algorithm works for directed graphs where bi-directional communication is not possible.

3. Problem formulation

In this paper, we focus on the dynamic estimation, where the estimated quantity is a weighted average of sensor inputs with different weights in a sensor network. Before stating the problem, we first provide motivating examples.

3.1. Motivating examples

3.1.1. Motivating Example 1: maximum-likelihood estimation

In a distributed network, say a sensor network, modelled by a directed graph, there is a reading $u_i(t)$ of the true value $r(t)$ perturbed by a noise v_i associated with the ith sensor:

$$u_i(t) = r(t) + v_i(t), \tag{3}$$

where $v_i(t)$ is assumed to have a zero mean σ_i^2 variance Gaussian distribution, $v_i(t)$ and $v_j(t)$ for $i \neq j$ are assumed to be independent. For the case $\sigma_i = \sigma_j$ for all i and j, the maximum-likelihood (ML) estimation of $r(t)$ is the direct average of $u_i(t)$ for all i, i.e.,

$$\hat{u}_{ML} = \frac{\sum_{i=1}^{n} u_i(t)}{n}. \tag{4}$$

For the case with $\sigma_i \neq \sigma_j$ for $i \neq j$, the ML estimation of $r(t)$, given the measurements $u_1(t), u_2(t), \ldots, u_n(t)$ with known $\sigma_1(t), \sigma_2(t), \ldots, \sigma_n(t)$, is the weighted average of $u_i(t)$, i.e.,

$$\hat{u}_{ML}(t) = \frac{\sum_{i=1}^{n} \beta_i u_i(t)}{\sum_{i=1}^{n} \beta_i}, \tag{5}$$

where $\beta_i = \frac{1}{\sigma_i^2} > 0$. Formulas (4) and (5) give ML estimations of u fusing all measurements in the network in the cases of identical and non-identical measurement accuracy, respectively. Traditionally, the implementation of formulas (4) and (5) needs all data relayed to a fusion centre, processed there, and then passed back to every sensors. While this scheme is not scalable to large networks, we focus on distributed solutions using consensus-based filter techniques.

3.1.2. Motivating Example 2: distributed cooperative sensing in a radio network

To demonstrate real-world applications of the weighted average estimation, here we mention distributed spectrum sensing for a radio network. Spectrum sensing is a fundamental problem in cognitive radio networks, which aims to improve spectrum utilisation by allowing unlicensed secondary users (such as smart phone users) to operate in the 'white space' of licensed spectrum bands without interfering licensed primary users (such as TV towers). Distributed cooperative sensing enables the secondary users to detect the presence of a primary user in the spectrum. Increasing attention has recently been paid to *distributed cooperative sensing* scheme (Yucek and Arslan 2009; Li, Yu, and Huang 2010), which uses the consensus algorithm to fuse sensor data over the network and to cooperatively estimate a common sensed variable (which is usually the detected energy level). However, due to imperfect channel conditions such as practical multi-path fading and shadowing effects, the equal gain combining (i.e., direct average) does not perform well as the algorithm considers equal weights for all sensor measurements. If channel conditions are considered, for example, the sinal-noise-ratio of the channel can be used to indicate accuracy of the measurement, in which case a weighted average consensus algorithm has shown improved performances (Zhang et al. 2011).

In this paper, we particularly discuss weighted average filtering algorithms, which lacks focused research attention in the past.

3.2. Control problem statement

In light of the motivating examples described above, we state our control problems in this subsection. We consider both the distributed average with identical and non-identical weights.

Problem 1 (distributed average estimation)**:** In a distributed sensor network modelled by a direct balanced graph, each sensor i has its measurement denoted as $u_i(t)$ that may be time-varying, design a distributed filter protocol to achieve $u_i(t) \to u_j(t) \to \frac{\sum_{i=1}^{n} u_i(t)}{n}$, $\forall i, j$, as $t \to \infty$.

Problem 2 (distributed weighted average estimation)**:** In a distributed sensor network modelled by a directed graph, each sensor i has its measurement $u_i(t)$ that may be time-varying, design a distributed filter protocol to achieve $u_i(t) \to u_j(t) \to \frac{\sum_{i=1}^{n} \beta_i u_i(t)}{\sum_{i=1}^{n} \beta_i}$, $\forall i, j$, as $t \to \infty$, where β_i is the pre-defined weight for the ith sensor.

Remark 3.1: In the above problem statement, Problem 2 extends Problems 1 in two aspects: (1) the identical weight average is extended to non-identical pre-defined weights and (2) the communication graph is extended from a balanced graph to a general directed graph. Note that the extension from average dynamic estimation to weighted average estimation is not trivial, as the generalisation from Problem 1 defined on a balanced graph to weighted averaging on a general directed graph defined in Problem 2 breaks the symmetric structure of the graph, which makes the theoretical analysis more challenging.

4. Distributed averaging with identical weights on a balanced graph

4.1. The protocol

In this section, we design a protocol to solve Problem 1. For a given network, whose topology can be described by the Laplacian matrix L constructed by the weight matrix $[w_{ij}]$, where $w_{ij} > 0$ if node i is connected to node j and $w_{ij} = 0$ otherwise, we propose the following protocol to solve the problem of distributed averaging with identical weights:

$$\begin{aligned}\dot{x}_i(t) &= -\gamma_1 \sum_{j\in\mathbb{N}(i)} w_{ij}(x_i(t) - x_j(t)) - \gamma_2(x_i(t) - u_i(t))\\ &\quad + \gamma_3 \sum_{j\in\mathbb{N}(i)} w_{ij}(\lambda_i(t) - \lambda_j(t)),\\ \dot{\lambda}_i(t) &= -\sum_{j\in\mathbb{N}(i)} w_{ij}(x_i(t) - x_j(t)),\end{aligned} \tag{6}$$

where $x_i(t)$ is the local estimation maintained by the ith sensor, $\lambda_i(t)$ is a co-state of the ith sensor, $u_i(t)$ is the input to the ith sensor, $\mathbb{N}(i)$ denotes the neighbour set of the ith sensor, and γ_1, γ_2 and γ_3 are positive constants. Note that the protocol (6) is distributed since the update of x_i and λ_i only needs information from the neighbouring sensors.

In a compact matrix form, the protocol writes

$$\dot{x} = -\gamma_1 L x - \gamma_2(x - u) + \gamma_3 L\lambda, \tag{7a}$$

$$\dot{\lambda} = -Lx, \tag{7b}$$

where x is the vector of estimation variables, $[x^T, \lambda^T]^T$ is the state of the filter system, L is the Laplacian matrix constructed by weights $[w_{ij}]$, and $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$ are coefficients.

Remark 4.1: Compared to the consensus filter (2) proposed by Freeman et al. (2006), which has L_I^T as a coefficient appearing in the dynamic equation, we replaced it with $\gamma_3 L$ in (7), so the update of x_i and λ_i requires information from the neighbouring set $\mathbb{N}(i)$ only. But the trade-off is that we need some gain conditions to guarantee the convergence of the protocol, which is discussed next.

4.2. Convergence analysis

In this subsection, by conducting convergence analysis, we prove that the protocol (6) provides a solution to Problem 1 defined in Section 3.1 under certain gain conditions. We first perform a state transformation in order to decouple the dynamic system (6) utilising graph properties of the network.

4.2.1. State transformation

Left multiplicating $\mathbf{1}^T$ on the dynamic of $\lambda(t)$ in the system Equation (7) generates zero on the right-hand side, which means $\mathbf{1}^T\lambda(t)$ is an invariant manifold in the evolution of the dynamic system. Motivated by this one-dimensional invariant property, we want to decouple the system dynamic into two parts: one part corresponding to this one-dimensional invariant manifold and the other corresponding to the rest of the dynamics.

As the equilibrium point of protocol (7) is not the origin, we first perform a translational transformation to shift the equilibrium point to the origin. Define

$$x^*(t) = \frac{\mathbf{1}\mathbf{1}^T u(t)}{n}, \tag{8}$$

$$\lambda^*(t) = \left(\begin{bmatrix}\gamma_3 L\\ \mathbf{1}^T\end{bmatrix}^T \begin{bmatrix}\gamma_3 L\\ \mathbf{1}^T\end{bmatrix}\right)^{-1} \begin{bmatrix}\gamma_3 L\\ \mathbf{1}^T\end{bmatrix}^T \times \begin{bmatrix}\gamma_2(\frac{\mathbf{1}\mathbf{1}^T}{n} - I)u(t)\\ \lambda_0\mathbf{1}^T\end{bmatrix}. \tag{9}$$

As a matrix inverse appears in the expression of $\lambda^*(t)$ above, we first show that the inverse exists. This result can be easily seen from the fact that L has rank $n-1$ for a connected balanced graph and $\mathbf{1}^T\gamma_3 L^T = \mathbf{0}$ meaning that the rows in $\gamma_3 L$ are orthogonal to $\mathbf{1}^T$.

Now we define the following translational transformation,

$$e(t) = \begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} = \begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} - \begin{bmatrix}x^*(t)\\ \lambda^*(t)\end{bmatrix}. \tag{10}$$

In the new coordinates, the dynamic of e is as follows:

$$\dot{e}(t) = Ae(t) - \begin{bmatrix}\dot{x}^*(t)\\ \dot{\lambda}^*(t)\end{bmatrix}, \tag{11}$$

where

$$A = \begin{bmatrix}-\gamma_1 L - \gamma_2 I & \gamma_3 L\\ -L & 0\end{bmatrix}, \tag{12}$$

$$\dot{x}^*(t) = \frac{\mathbf{1}\mathbf{1}^T\dot{u}(t)}{n}, \tag{13}$$

$$\dot{\lambda}^*(t) = \left(\begin{bmatrix}\gamma_3 L\\ \mathbf{1}^T\end{bmatrix}^T \begin{bmatrix}\gamma_3 L\\ \mathbf{1}^T\end{bmatrix}\right)^{-1} \begin{bmatrix}\gamma_3 L\\ \mathbf{1}^T\end{bmatrix}^T \times \begin{bmatrix}\gamma_2(\frac{\mathbf{1}\mathbf{1}^T}{n} - I)\dot{u}(t)\\ \lambda_0\mathbf{1}^T\end{bmatrix}. \tag{14}$$

After the equilibrium point shifting, we are able to transform the system into the decoupled space with an additional similarity transformation. About this point, we have the following proposition.

Proposition 4.1: *For a connected balanced graph, there exists a similarity transformation, such that the average consensus error* $e_1(t) = x(t) - x^*(t) = x(t) - \frac{\mathbf{1}\mathbf{1}^T u(t)}{n}$ *of the system* (6) *with the input* $u(t)$ *can be expressed as follows:*

$$\begin{aligned} \dot{z}_2(t) &= Mz_2(t) - Q_1\dot{x}^*(t) - Q_2\dot{\lambda}^*(t), \\ e_1(t) &= Q_1^T z_2(t), \end{aligned} \tag{15}$$

where $z_2(t) \in \mathbb{R}^{2n-1}$, $Q_1 \in \mathbb{R}^{(2n-1)\times n}$, $Q_2 \in \mathbb{R}^{(2n-1)\times n}$, *and* $M \in \mathbb{R}^{(2n-1)\times(2n-1)}$. *The eigenvalues of* M *are identical to the non-zero eigenvalues of the* $(2n-1) \times (2n-1)$ *matrix* A *defined in* (12).

Proof: See Appendix A. □

Remark 4.2: For time-invariant input $u(t) = u(0)$ for all $t > 0$, the weighted average consensus error e_1 has the following expression:

$$\dot{z}_2 = Mz_2, \; e_1 = Q_1^T z_2 \tag{16}$$

as both $\dot{x}^*$ and $\dot{\lambda}^*$ equal to zero in this case.

4.2.2. Convergence conditions

Next, we provide convergence conditions of (15). Note that for the protocol (1), convergence is guaranteed for any positive gains under some mild conditions (Freeman et al. 2006). However, in our case, we need extra conditions for (6) to guarantee its convergence since the modification from (1) to (6) breaks the symmetric property of the protocol (1). On this point, we have the following proposition.

Proposition 4.2: *For the system* (6) *running on a connected balanced graph with time-varying input signal* $u(t)$*, if all non-zero eigenvalues of the matrix* A *defined in* (12) *locate on the left-half-plane* (*LHP*) *strictly, the sensor state* x_i *for all* i *tracks the average of the input* $u(t)$*, which is* $\frac{\mathbf{1}\mathbf{1}^T u(t)}{n}$*, with a bounded tracking error. Particularly, for time-invariant input signal* $u(t) = u(0)$ *for* $t > 0$*, this condition guarantees that the tracking error converges to zero for all sensors.*

Proof: See Appendix B. □

To meet the condition that all non-zero eigenvalues of the matrix A strictly locates on the LHP often results in solving linear matrix inequalities, which is often computationally expensive especially for large sensor networks. To avoid this difficulty, we give a sufficient gain condition as stated in the following theorem.

Theorem 4.3: *The protocol* (6) *solves Problem* 1 *defined on a connected balanced graph with a bounded error provided the following gain conditions hold,*

$$\gamma_3 > 0, \tag{17a}$$

$$\gamma_2 > \max\left\{\max_{i\in\mathbb{S}}\left(4\gamma_3\left(b_i^2 - a_i^2\right)\right),\, 0\right\}, \tag{17b}$$

$$\gamma_1 > \max \times \left\{\max_{i\in\mathbb{S}}\left(\frac{-a_i^2\gamma_2 + b_i\sqrt{4a_i^4\gamma_3 + 4a_i^2 b_i^2\gamma_3 + b_i^2\gamma_2^2}}{a_i(a_i^2 + b_i^2)}\right),\, 0\right\}, \tag{17c}$$

where a_i, b_i *denote the real and imaginary part of the* i*th eigenvalue of the matrix* L *and* $\mathbb{S} = \{2, 3, \ldots, n\}$. *That is, the sensor state* x_i *in* (6) *for all* i *tracks the average of the time-varying input signal* $u(t)$, $\frac{\mathbf{1}\mathbf{1}^T u(t)}{n}$, *with a bounded tracking error. In addition, the tracking error converges to zero for all sensors in the case of time-invariant input signal* $u(t) = u(0)$ *for* $t > 0$.

Proof: Based on the conclusion drawn in Proposition 4.2, we can prove the result by showing that the non-zero eigenvalues of the matrix A defined in (12) locate on LHP strictly. To this end, we directly solve the eigenvalues of this matrix. That is, we let

$$\det\left(\begin{bmatrix} -\gamma_1 L - \gamma_2 I - \mu I & \gamma_3 L \\ -L & -\mu I \end{bmatrix}\right) = 0, \tag{18}$$

where μ represents the eigenvalue. Noticing that the off-diagonal blocks commute, we equivalently get

$$\det(\gamma_3 L^2 + \mu^2 I + \mu\gamma_2 I + \mu\gamma_1 L) = 0. \tag{19}$$

We have $L\mathbf{1} = 0$ and $\mathbf{1}^T L = 0$ for balanced graphs. The characteristic Equation (19) is a polynomial of L, so we get $(\gamma_3\eta^2 + \mu^2 I + \mu\gamma_2 + \mu\gamma_1\eta) = 0$ with η denoting the eigenvalue of L. Solving this equation yields

$$\mu_i = \frac{-(\gamma_2 + \gamma_1\eta_i) \pm \sqrt{(\gamma_2 + \gamma_1\eta_i)^2 - 4\gamma_3\eta_i^2}}{2}. \tag{20}$$

Since η is the eigenvalue of the Laplacian matrix L, we know $\eta_1 = 0$ and $Re(\eta_i) > 0$ for $i = 2, 3, \ldots, n$. According to (20), we have $\mu_1 = 0$, which is in correspondence to $\eta_1 = 0$.

Up to now, we have find the expressions of μ_i and also find the zero eigenvalue $\mu_1 = 0$. Hereafter, we prove that the rest eigenvalues $\mu_2, \mu_3, \ldots, \mu_n$ as given in (20), locate on LHP, i.e., $\mathrm{Re}(\mu_i) < 0$ for $i = 2, 3, \ldots, n$.

Let $a_i = \mathrm{Re}(\eta_i)$ and $b_i = \mathrm{Im}(\eta_i)$, then $\eta_i = a_i + b_i\sqrt{-1}$. For notational convenience, we define c_i and d_i as follows:

$$c_i \triangleq \mathrm{Re}\big((\gamma_2+\gamma_1\eta_i)^2 - 4\gamma_3\eta_i^2\big) = (\gamma_2+\gamma_1 a_i)^2 - \gamma_1^2 b_i^2 - 4\gamma_3 a_i^2 + 4\gamma_3 b_i^2, \tag{21a}$$

$$d_i \triangleq \mathrm{Im}\big((\gamma_2+\gamma_1\eta_i)^2 - 4\gamma_3\eta_i^2\big) = 2\gamma_1 b_i(\gamma_2+\gamma_1 a_i) - 8a_i b_i\gamma_3. \tag{21b}$$

With these notations, we have $\cos(\theta_i) = \frac{c_i}{\sqrt{c_i^2+d_i^2}}$, where θ_i denoting the argument of the complex number $(\gamma_2 + \gamma_1\eta_i)^2 - 4\gamma_3\eta_i^2$. Also we have $\mathrm{Re}(\sqrt{(\gamma_2+\gamma_1\eta_i)^2-4\gamma_3\eta_i^2}) = \pm\sqrt{c_i^2+d_i^2}\cos\frac{\theta_i}{2} = \pm\sqrt{c_i^2+d_i^2}\sqrt{\frac{\cos(\theta_i)+1}{2}} = \pm\sqrt{\frac{c_i+\sqrt{c_i^2+d_i^2}}{2}}$.

To prove $\mathrm{Re}(\mu_i) < 0$ for $i = 2, 3, \ldots, n$, the following reasoning is used:

$$\begin{aligned}\mathrm{Re}(\mu_i) < 0 &\Leftarrow \mathrm{Re}(\gamma_2+\gamma_1\eta_i) > \sqrt{\frac{c_i+\sqrt{c_i^2+d_i^2}}{2}} \\ &\Leftarrow 2(\gamma_2+\gamma_1 a_i)^2 > c_i + \sqrt{c_i^2+d_i^2} \\ &\Leftarrow 2(\gamma_2+\gamma_1 a_i)^2 - c_i > \sqrt{c_i^2+d_i^2} \\ &\Leftarrow \begin{cases} \big(2(\gamma_2+\gamma_1 a_i)^2 - c_i\big)^2 - c_i^2 - d_i^2 > 0 \\ 2(\gamma_2+\gamma_1 a_i)^2 - c_i \geq 0 \end{cases}.\end{aligned} \tag{22}$$

To prove $\big(2(\gamma_2+\gamma_1 a_i)^2 - c_i\big)^2 - c_i^2 - d_i^2 > 0$ (the first line in (22)), we have the following by substituting the expression of c_i and d_i inside:

$$\begin{aligned}&\big(2(\gamma_2+\gamma_1 a_i)^2 - c_i\big)^2 - c_i^2 - d_i^2 > 0 \\ &\Leftarrow a_i^4\gamma_1^2\gamma_3 + 2a_i^3\gamma_1\gamma_2\gamma_3 + a_i^2 b_i^2\gamma_1^2\gamma_3 \\ &\quad - 4a_i^2 b_i^2\gamma_3^2 + a_i^2\gamma_2^2\gamma_3 - b_i^2\gamma_2^2\gamma_3 > 0.\end{aligned} \tag{23}$$

The left-hand side of the above inequality can be regarded as a function of γ_1 and the function $f(\gamma_1) = a_i^4\gamma_1^2\gamma_3 + 2a_i^3\gamma_1\gamma_2\gamma_3 + a_i^2 b_i^2\gamma_1^2\gamma_3 - 4a_i^2 b_i^2\gamma_3^2 + a_i^2\gamma_2^2\gamma_3 - b_i^2\gamma_2^2\gamma_3$ is a convex quadratic form with respect to γ_1 and $f(\gamma_1) = 0$ always has solutions because the discriminant of $f(\gamma_1)$ is $4a_i^2 b_i^2\gamma_3^2(4a_i^4\gamma_3 + 4a_i^2 b_i^2\gamma_3 + b_i^2\gamma_2^2) \geq 0$. A sufficient condition for $f(\gamma_1) > 0$ is to choose γ_1 greater than the larger root of $f(\gamma_1^*) = 0$, i.e.,

$$\gamma_1 > \gamma_1^* = \frac{-a_i^2\gamma_2 + b_i\sqrt{4a_i^4\gamma_3 + 4a_i^2 b_i^2\gamma_3 + b_i^2\gamma_2^2}}{a_i(a_i^2+b_i^2)}. \tag{24}$$

This inequality holds true under the condition $\gamma_1 > \max\big\{\max_{i\in\mathbb{S}}\big(\frac{-a_i^2\gamma_2+b_i\sqrt{4a_i^4\gamma_3+4a_i^2b_i^2\gamma_3+b_i^2\gamma_2^2}}{a_i(a_i^2+b_i^2)}\big),\ 0\big\}$.

We prove $2(\gamma_2 + \gamma_1 a_i)^2 - c_i \geq 0$ (the second line in (22)) by substituting the expression of c_i into the inequality as follows:

$$2(\gamma_2+\gamma_1 a_i)^2 - c_i \geq 0 \Leftarrow (\gamma_2+\gamma_1 a_i)^2 + \gamma_1^2 b_i^2 + 4\gamma_3 a_i^2 - 4\gamma_3 b_i^2 \geq 0. \tag{25}$$

With the condition $\gamma_2 > \max\{\max_{i\in\mathbb{S}}(4\gamma_3(b_i^2 - a_i^2)),\ 0\} \geq 4\gamma_3(b_i^2 - a_i^2)$ and the fact $\gamma_1^2 a_i^2 + 2\gamma_2\gamma_1 a_i + \gamma_1^2 b_i^2 \geq 0$ for $\gamma_2 > 0$ and $\gamma_1 > 0$, we conclude $(\gamma_2^2 + 4\gamma_3 a_i^2 - 4\gamma_3 b_i^2) + (\gamma_1^2 a_i^2 + 2\gamma_2\gamma_1 a_i + \gamma_1^2 b_i^2) \geq 0$ is true for $i = 2, 3, \ldots, n$.

Up to now, we have proved the inequalities in (22) hold, which suffices $\mathrm{Re}(\mu_i) < 0$ for $i = 2, 3, \ldots, n$. According to the reasoning at the beginning of this proof, $\mathrm{Re}(\mu_i) < 0$ for $i = 2, 3, \ldots, n$, reaches the conclusion. This completes the proof. □

Note that the gain condition in Theorem 4.3 is a sufficient one for the convergence of the protocol (6).

Remark 4.3: To guarantee the convergence, the parameters γ_1, γ_2, and γ_3 in protocol (6) can be chosen using the following procedure according to Theorem 4.3: step 1: assign γ_3 a positive constant; step 2: compute the lower bound $\max\{\max_{i\in\mathbb{S}}(4\gamma_3(b_i^2 - a_i^2)),\ 0\}$ for γ_2, and assign γ_2 a value greater than this lower bound; step 3: compute the lower bound $\max\{\max_{i\in\mathbb{S}}\big(\frac{-a_i^2\gamma_2+b_i\sqrt{4a_i^4\gamma_3+4a_i^2b_i^2\gamma_3+b_i^2\gamma_2^2}}{a_i(a_i^2+b_i^2)}\big),\ 0\}$ for γ_1, and assign γ_1 a value greater than this lower bound.

Remark 4.4: The gain condition in Theorem 4.3 requires global information on the graph topology. For some large scale networks, the eigenvalues often demonstrate certain statistical properties (Goh, Kahng, and Kim 2001), which can be employed for the design of the gains γ_1, γ_2, and γ_3. Also, it can be derived that a conservative choice of the gains with $\gamma_2 > 1$ and $\gamma_1 \gg \gamma_2 \gg \gamma_3 > 0$ suffices the gain condition. Note that the recent work by Qin, Zheng, and Gao (2011) and the work by Qin and Gao (2012) simplify the gain conditions for the consensus problem with second-order dynamics, which will be explored in our future work for a possible simplification of the gain conditions.

5. Dynamic estimation of weighted average on a directed graph

In this section, we design a protocol to solve Problem 2. We first consider the weighted averaging on a balanced graph in a distributed manner, and then extend the result to general directed graphs. For a given balanced network, we propose the following protocol to solve the distributed dynamic estimation of the weighted average:

$$\begin{aligned}\dot{x}_i(t) &= -\gamma_1\epsilon_{i1}\sum_{j\in\mathbb{N}(i)} w_{ij}(x_i(t)-x_j(t)) - \gamma_2(x_i(t)\\ &\quad - u_i(t)) + \gamma_3\epsilon_{i1}\sum_{j\in\mathbb{N}(i)} w_{ij}(\lambda_i(t)-\lambda_j(t)),\\ \dot{\lambda}_i(t) &= -\epsilon_{i1}\sum_{j\in\mathbb{N}(i)} w_{ij}(x_i(t)-x_j(t)),\end{aligned} \tag{26}$$

where $x_i(t)$ is the local estimation]maintained by the ith sensor, $\lambda_i(t)$ is a co-state of the ith sensor, u_i is the input to the ith sensor, $\epsilon_{i1} = \frac{1}{\beta_i}$ and β_i is the pre-defined weight of the ith sensor. $\mathbb{N}(i)$ denotes the neighbour set of the ith sensor. γ_1, γ_2, and γ_3 are positive constants. Note that the protocol (26) is distributed since the update of $x_i(t)$ and $\lambda_i(t)$ in the protocol only need information from the neighbouring sensors.

In a compact matrix form, the protocol writes

$$\begin{aligned}\dot{x}(t) &= -\gamma_1\Lambda_1 Lx(t) - \gamma_2(x(t)-u(t)) + \gamma_3\Lambda_1 L\lambda(t),\\ \dot{\lambda}(t) &= -\Lambda_1 Lx(t),\end{aligned} \tag{27}$$

where $\Lambda_1 = \mathrm{diag}([\epsilon_{11}, \epsilon_{21}, \ldots, \epsilon_{n1}])$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$ are constant coefficients.

Remark 5.1: Compared with Protocol (7), the desired weight matrix Λ_1 shows up in (27) as the coefficient of L. The convergence proof of (27) is not straightforward due to the presence of Λ_1. Although a similar procedure can be followed for the proof, the transformation matrices used in the direct average case is not applicable to (27). In the next, we present a different transformation and conduct convergence analysis of the proposed protocol.

5.1. *Convergence analysis*

Protocol (26) provides a solution for Problem 2 defined on a balanced graph. In this section, we give the following gain condition guaranteeing the convergence of this protocol.

Theorem 5.1: *Protocol* (26) *solves Problem* 2 *defined on a connected balanced graph with a bounded error, provided the following gain conditions hold*

$$\gamma_3 > 0, \tag{28a}$$

$$\gamma_2 > \max\left\{\max_{i\in\mathbb{S}}\left(4\gamma_3\left(b_i^2 - a_i^2\right)\right),\ 0\right\}, \tag{28b}$$

$$\gamma_1 > \max \times \left\{\max_{i\in\mathbb{S}}\left(\frac{-a_i^2\gamma_2 + b_i\sqrt{4a_i^4\gamma_3 + 4a_i^2b_i^2\gamma_3 + b_i^2\gamma_2^2}}{a_i(a_i^2+b_i^2)}\right),\ 0\right\}, \tag{28c}$$

where a_i, b_i denote the real and imaginary part of the ith eigenvalue of the matrix $\Lambda_I L$ and $\mathbb{S} = \{2, 3, \ldots, n\}$. That is, the sensor state x_i in (26) *for all i tracks the weighted average of the input, $\frac{\mathbf{1}\beta^T u(t)}{\beta^T\mathbf{1}}$, with a bounded tracking error. In addition, the tracking error converges to zero for all sensors in the case of time-invariant input signal $u(t) = u(0)$ for $t > 0$.*

Proof: See Appendix C. ☐

5.2. *Special case: weighted averaging on undirected graphs*

For an undirected graph, we have $L = L^T$, which means if a sensor can receive information from another one, it must be able to send information to that one. Due to the special structure, system (26) in this case converges for all positive γ_1, γ_2, and γ_3, as stated in the following corollary.

Corollary 5.2: *For the system* (6) *with $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$ running on a connected undirected graph with time-varying input signal $u(t)$, the sensor state x_i for all i tracks the weighted average of the input $u(t)$, $\frac{\mathbf{1}\beta^T u(t)}{\beta^T\mathbf{1}}$, with a bounded tracking error. Particularly in the case of time-invariant input signal $u(t) = u(0)$ for $t > 0$, the tracking error converges to zero for all sensors.*

Proof: See Appendix D. ☐

5.3. *Extension: weighted averaging on general directed graphs*

A connected directed balanced graph is a special case of directed graphs with strongly connected topology. In this part, we extend the above results to general directed graphs for weighted averaging.

The proposed protocol in this paper for solving weighted averaging on general directed graphs is as follows:

$$\begin{aligned}\dot{x}_i(t) &= -\gamma_1\epsilon_{i2}\sum_{j\in\mathbb{N}(i)} w_{ij}(x_i(t)-x_j(t)) - \gamma_2(x_i(t)\\ &\quad - u_i(t)) + \gamma_3\epsilon_{i2}\sum_{j\in\mathbb{N}(i)} w_{ij}(\lambda_i(t)-\lambda_j(t)),\\ \dot{\lambda}_i(t) &= -\epsilon_{i2}\sum_{j\in\mathbb{N}(i)} w_{ij}(x_i(t)-x_j(t)),\end{aligned} \tag{29}$$

where $x_i(t)$ is the local estimation maintained by the ith sensor, $\lambda_i(t)$ is a co-state of the ith sensor, $u_i(t)$ is the input to the ith sensor, $\epsilon_{i2} = \frac{\alpha_i}{\beta_i}$, with $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ denotes the left zero eigenvalue of L, β_i is the pre-defined weigh constant for sensor i, γ_1, γ_2 and γ_3 are positive constants.

In a compact matrix form, the protocol writes

$$\dot{x}(t) = -\gamma_1 \Lambda_2 L x(t) - \gamma_2 (x(t) - u(t)) + \gamma_3 \Lambda_2 L \lambda,$$
$$\dot{\lambda}(t) = -\Lambda_2 L x(t), \tag{30}$$

where $\Lambda_2 = \mathrm{diag}([\epsilon_{12}, \epsilon_{22}, \ldots, \epsilon_{n2}]) = \mathrm{diag}(\alpha)\mathrm{diag}^{-1}(\beta)$ with $\epsilon_i = \sigma_i^2$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$ are constant coefficients.

Note that $\Lambda_2 L = \mathrm{diag}(\alpha)\mathrm{diag}^{-1}(\beta) L = \mathrm{diag}^{-1}(\beta)\mathrm{diag}(\alpha) L$ and the matrix $\mathrm{diag}(\alpha)L$ is a Laplancian matrix with left zero eigenvector at $\mathbf{1}$ since $\mathbf{1}^T \mathrm{diag}(\alpha) L = \alpha^T L = 0$. By regarding $\mathrm{diag}(\alpha)L$ as a new Laplacian matrix, protocol (30) reduce to the weighted average protocol on a balanced graph.

We have the following theorem.

Theorem 5.3: *The protocol* (29) *solves Problem* 2 *defined on a strongly connected directed graph with a bounded tracking error, provided the following gain conditions hold:*

$$\gamma_3 > 0, \tag{31a}$$

$$\gamma_2 > \max\left\{\max_{i\in\mathbb{S}} \left(4\gamma_3 (b_i^2 - a_i^2)\right),\ 0\right\}, \tag{31b}$$

$$\gamma_1 > \max \times \left\{ \max_{i \in \mathbb{S}} \left(\frac{-a_i^2 \gamma_2 + b_i \sqrt{4a_i^4 \gamma_3 + 4a_i^2 b_i^2 \gamma_3 + b_i^2 \gamma_2^2}}{a_i (a_i^2 + b_i^2)} \right), 0 \right\}, \tag{31c}$$

where a_i, b_i denote the real and imaginary part of the ith eigenvalue of the matrix $\Lambda_2 L$ and $\mathbb{S} = \{2, 3, \ldots, n\}$. That is, the sensor state $x_i(t)$ in (30) *for all i tracks the weighted average, $\frac{\mathbf{1}\beta^T u(t)}{\beta^T \mathbf{1}}$, with a bounded tracking error. In addition, the tracking error converges to zero for all sensors for time-invariant input signal $u(t) = u(0)$ for $t > 0$.*

The proof of Theorem 5.3 is similar to that of Theorem 5.1 and is omitted here.

Remark 5.2: Note that the requirement of strongly connected graph is necessary to guarantee that $\mathrm{diag}(\alpha)L$ is a Laplacian matrix of a connected balanced graph, to which Theorem 5.1 applies.

Remark 5.3: It is worth noting that gain conditions are obtained to guarantee convergence for the proposed averaging protocols to solve Problems 1 and 2. However, although the Equations (17), (28) and (31) have the same form, the conditions are different in the sense that a_i and b_i are defined differently. Specifically, in (17), a_i and b_i are defined as the real and imaginary parts of L; in (28), they are defined as the real and imaginary parts of $\Lambda_1 L$; in (31), they are defined as the real and imaginary parts of $\Lambda_2 L$.

Remark 5.4: The gain condition (31) for general directed graphs requires knowledge on the spectrum of the graph Laplacian matrix, which is in contrast to the undirected graph case without such a requirement (see Section 5.2). This is because that a general directed graph does not have a symmetric topological structure and the spectrum of the matrix $\Lambda_2 L$ shows up as components in the eigenvalues of the filter's system matrix. Same as work by Ren and Atkins (2007) and Ren (2008), prior knowledge on the graph topology is needed. The results applies to applications where fixed topologies are prior known such as in power grids (Bai, Wei, Fujisawa, and Wang 2008) and in mobile robot network with maintained connectivity (Michael, Zavlanos, Kumar, and Pappas 2008).

6. Simulation examples

6.1. Simulation set-ups

We conduct simulations to compare the performances of the proposed distributed filter algorithms on the Motivating Example 1. We consider a small network (as shown in Figure 1(a)) represented by the following Laplacian matrix:

$$L_A = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}. \tag{32}$$

Clearly, this is a balanced graph since the in-degree equals the out-degree for all nodes on the graph. By introducing an extra link from node 5 to node 3, the balanced structure is destroyed and the new graph, as shown in Figure 1(b), becomes unbalanced. The Laplacian matrix of the new graph is as follows by assigning each existing link with an unit

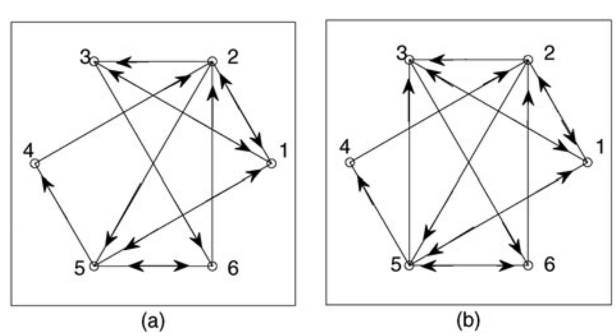

Figure 1. Graph topologies used in the simulations: (a) a balanced graph and (b) an imbalanced directed graph.

weight:

$$L_B = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}. \qquad (33)$$

6.2. *Simulation 1: distributed average estimation on a balanced graph*

In this section, we first show the simulation results with the proposed protocol (6) to solve distributed average estimation on a balanced graph. Then, the proposed protocol is compared with existing protocols on solving the same problem.

6.2.1. *Performance of the proposed protocol*

In this part, we show simulation results with the proposed protocol (6) to solve distributed average estimation on the balanced graph shown in Figure 1(a), which is associated with the Laplacian matrix L_A in (32). Table 1 summarises some parameters used in the simulation. We choose parameters of the filter according to Theorem 4.3.

Table 1. Parameters used in Simulation 1, where a_i and b_i are the ith real part and the ith imaginary part of the graph Laplacian matrix L_A, respectively.

Node i	1	2	3	4	5	6
a_i	0	1.382	2	3	3.6180	4
b_i	0	0	0	0	0	0

According to (17), we first choose $\gamma_3 = 1$. Then, with the value of γ_3, we get $\max\left\{\max_{i\in\mathbb{S}}\left(4\gamma_3\left(b_i^2 - a_i^2\right)\right),\ 0\right\} = 0$, thus let $\gamma_2 = 1$. With both the values of γ_2 and γ_3, we get $\max\left\{\max_{i\in\mathbb{S}}\left(\frac{-a_i^2\gamma_2 + b_i\sqrt{4a_i^4\gamma_3 + 4a_i^2b_i^2\gamma_3 + b_i^2\gamma_2^2}}{a_i(a_i^2+b_i^2)}\right),\ 0\right\} = 0$ and the selection of $\gamma_1 = 1$ suffices the gain inequality (17). For the time-invariant input case, we use the input u =[1,2,3,4,5,6] in the simulation and initialise both x and λ randomly. As shown in Figure 2(a), by running the protocol (6), the decision variable x_i for all sensors converges to the ideal average of the constant input. Note that x_i approaches consensus as time elapses, while the co-state λ_i converges to different values instead of a common value. This observation is in consistent with the steady state expression of λ^* in (8), which generally has different values in its entries.

To test the effectiveness of the proposed protocol for tracking the average of dynamic signals, we consider the time-varying input $u(t) = [10\sin(0.8t + 0.3),$

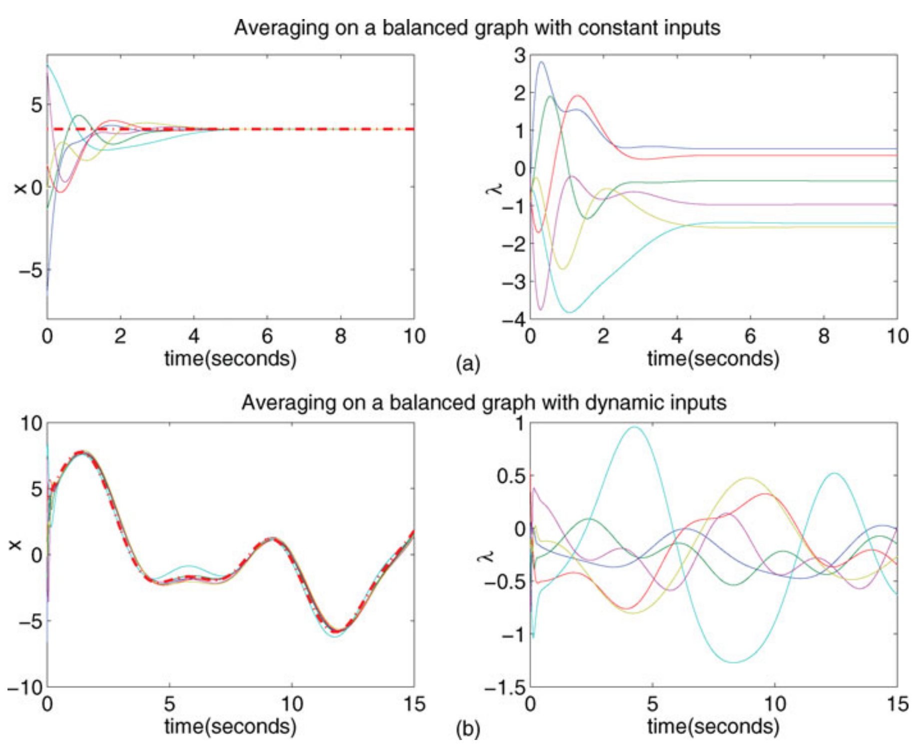

Figure 2. Averaging on a balanced graph with (a) constant inputs and (b) dynamic inputs. The red dashed line represents the ideal average.

5sin (1.6t + 0.6), 13sin (0.4t + 0.9), 14sin (0.8t + 1.2), −9sin (1.6t + 1.5), 7sin (0.4t + 0.8)]. With $\gamma_1 = \gamma_2 = 10$, $\gamma_3 = 100$, which satisfy the gain conditions (17) given in Theorem 4.3, and the same set-up for other parameters as in the constant input case, the proposed protocol tracks the desired average of the time-varying input with a bounded error as shown in Figure 2(b). For the co-state λ, it keeps changing to track the $\lambda^*(t)$ in (8), which varies with time as $u(t)$ is time-varying.

6.2.2. *Comparison with existing methods*

6.2.2.1. Comparison with the low-pass consensus filter by Olfati-Saber and Shamma (2005). In this part, we compare the proposed protocol (6) with the low-pass consensus filter proposed by Olfati-Saber and Shamma (2005) on the estimation accuracy. As shown in Equation (7) in the paper by Olfati-Saber and Shamma (2005), the filter writes $\dot{x} = -(I + \Delta + L)x + (I + A)u$ in a compact form, where I represents an identity matrix, A is the adjacency matrix, L is the Laplacian matrix, Δ is the diagonal matrix consisting of the diagonal elements of the Laplacian matrix. The comparison is performed with $L = L_A$ for the low-pass consensus filter. For the same constant input $u = [1, 2, 3, 4, 5, 6]$ and the same initialisation on x, the output of the low-pass consensus filter cannot reach a common value as shown in Figure 3, which is in contrast to the convergence results by the proposed method shown in Figure 2(a).

6.2.2.2. Comparison with the PI Consensus Filter by Freeman et al. (2006). In addition, we compare the proposed protocol (6) with the PI average consensus filter protocol (1) proposed by Freeman et al. (2006) on the communication load expenses when conducting averaging task on the graph shown in Figure 1(a). It is noteworthy that protocol (1) fails if message routing is not allowed in the network since the requirement of protocol (1) on bi-directional information exchanges cannot be satisfied by the directed graph shown in Figure 1(a). To make the comparison, we assume that message routings are allowed for protocol (1) and the shortest path is taken to route the message. We measure the communication load by the number of one-hop communications employed for the update of variables in each step. For protocol (1), the update of x_3, which is associated with node 3, requires information on $x_1(t)$ from node 1 and $x_2(t)$ from node 2 to account for the term $\sum_{j\neq 3} a_{3j}(t)(x_3(t) - x_j(t))$ in (1) and also requires information on $\lambda_6(t)$ from node 6 to account for the term $\sum_{j\neq 3} b_{j3}(t)(\lambda_3(t) - \lambda_j(t))$. Note that $a_{3j}(t)$ is associated with the link from node 3 to node j, and $a_{31}(t)$ and $a_{32}(t)$ are non-zero. $b_{j3}(t)$ is associated with the link from node j to node 3 and $b_{63}(t)$ is non-zero. Remarkably, there is no direct connection from node 6 to node 3 and one of the shortest route available from node 6 to node 3 is from node 6 to node 2 and then to node 3 (totally two one-hop communications). Therefore, the minimum total number of one-hop communications required to update $x_3(t)$ in each step is 4. The update of $\lambda_3(t)$ re-uses information on $x_1(t)$ and $x_2(t)$ to account for $\sum_{j\neq 3} b_{3j}(t)(x_3(t) - x_j(t))$, thus does not add new expenses on the communication load. With a similar analysis, we can conclude that for the small graph shown in Figure 2(a), at least totally 32 one-hop communications are required to update $x(t)$ and $\lambda(t)$ for all nodes in each step by taking the shortest routing path. In contrast, the communication topology for protocol (6) coincides with the graph topology and requires totally 28 one-hop communications to update both $x(t)$ and $\lambda(t)$ in (6). In this regard, the communication load is 32 for protocol (1) vs. 28 for protocol (6), which demonstrates the advantage of the proposed protocol in reducing communication burdens.

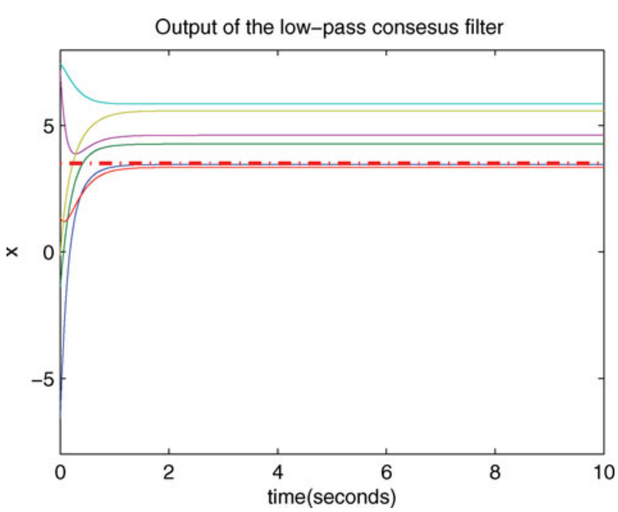

Figure 3. Output of the low-pass consensus filter for averaging on a balanced graph with constant inputs. The red dashed line represents the ideal average.

6.3. *Simulation 2: distributed weighted average estimation on a balanced graph*

In this part, we show simulation results with the proposed protocol (26) to solve distributed weighted average estimation on the balanced graph shown in Figure 1(a). It is assumed in this simulation that each sensor has a standard deviation σ_i and therefore has the desired weights $\beta_i = 1/\sigma_i^2$. The values of σ_i and β_i are summarised in Table 2. For this set-up, the values of a_i and b_i, which are the ith real part and the ith imaginary part of $\Lambda_1 L_A = \mathrm{diag}^{-1}(\beta)L_A$, are also shown in Table 2. We choose $\gamma_1 = \gamma_2 = \gamma_3 = 1$, which satisfy the gain conditions (28) given in Theorem 5.1 for the constant input case. With the same set of static inputs as in Section 6.2, the decision variable x in (26) converges to the ideal average weighted by β as shown in Figure 4(a) while the co-state λ_i converges to different values instead of a common value. With the same time-varying input as

Table 2. Parameters used in Simulation 2, where σ_i denotes the standard deviation of the ith sensor; $\beta_i = 1/\sigma_i^2$; a_i and b_i are the ith real part and the ith imaginary part of $\mathrm{diag}^{-1}(\beta)L_A$, respectively.

Node i	1	2	3	4	5	6
σ_i	1	1.5	2	2.5	3	3.5
β_i	1.0000	0.4444	0.2500	0.1600	0.1111	0.0816
a_i	0	35.9470	15.4843	5.0156	9.5332	9.5332
b_i	0	0	0	0	0.3490	−0.3490

in Section 6.2, the time history of x and λ by running the protocol (26) under $\gamma_1 = \gamma_2 = 10$, $\gamma_3 = 100$, which also satisfy the gain conditions given in Theorem 5.1, is shown in Figure 4(b). It can be observed that x tracks the desired weighted average of the time-varying signal with a bounded error while the co-state variable keeps changing to track the desired time varying $\lambda^*(t)$ in (8).

6.4. Simulation 3: distributed weighted average estimation on a general directed graph

In this part, we show simulation results with the proposed protocol (29) to solve distributed weighted average estimation on a general directed graph as shown in Figure 1(b) with the Laplacian matrix L_B in (33). For this graph, the left zero eigenvector of its Laplacian matrix can be computed as shown in Table 3. We use the same desired weight β as in Section 6.3. For this set-up, a_i and b_i, which are the ith real part and the ith imaginary part of $\Lambda_2 L_B = \mathrm{diag}(\alpha)\mathrm{diag}^{-1}(\beta)L_B$, are as summarised in Table 3. It can be validated that $\gamma_1 = \gamma_2 = \gamma_3 = 1$ satisfy the gain conditions (31) given in Theorem 5.3. With the same set of constant inputs as in Section 6.2, the decision variable of x in protocol (29) converges to the ideal average weighted by β as shown in Figure 5(a). The co-state λ converges to different values as expected. With the same time-varying input as in Section 6.2, the time history of x and λ by running the protocol (29) using $\gamma_1 = \gamma_2 = 10$, $\gamma_3 = 100$, is shown in Figure 5(b). It can be observed that x tracks the desired weighted average of the time-varying signal with a bounded error while the co-state variable keeps changing to track the desired time varying $\lambda^*(t)$ in (8).

Table 3. Parameters used in Simulation 3, where σ_i denotes the standard deviation of the ith sensor; $\beta_i = 1/\sigma_i^2$; a_i and b_i are the ith real part and the ith imaginary part of $\mathrm{diag}(\alpha)\mathrm{diag}^{-1}(\beta)L_B$, respectively.

Node i	1	2	3	4	5	6
α_i	0.3932	0.3932	0.2808	0.3932	0.5055	0.4493
β_i	1.0000	0.4444	0.2500	0.1600	0.1111	0.0816
a_i	0	17.1925	7.9586	1.9831	3.5945	3.5945
b_i	0	0	0	0	0.0567	−0.0567

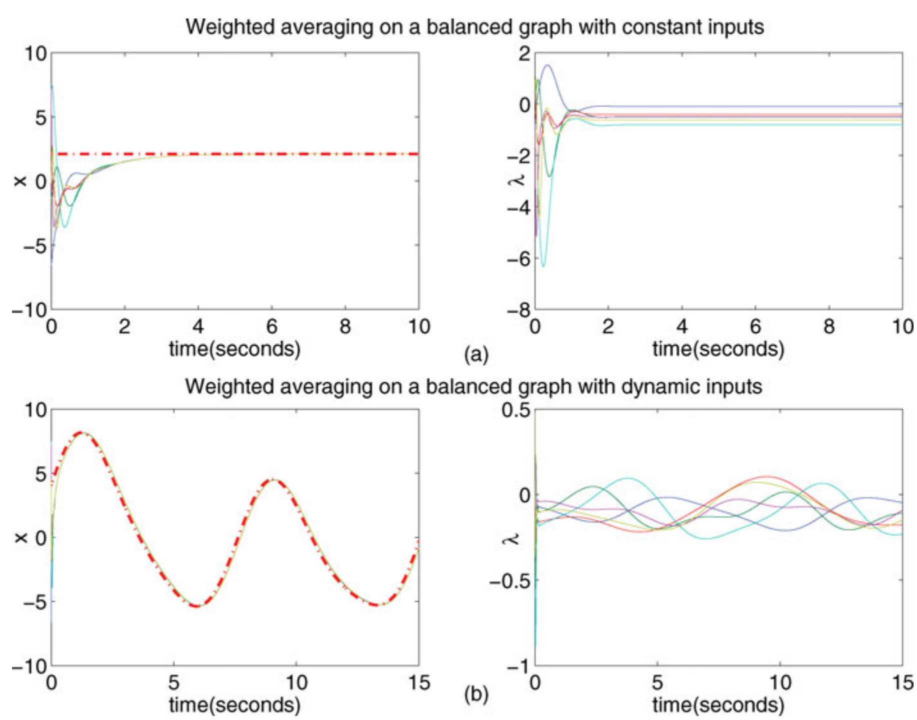

Figure 4. Weighted averaging on a balanced graph with (a) constant inputs and (b) dynamic inputs. The red dashed line represents the ideal weighted average.

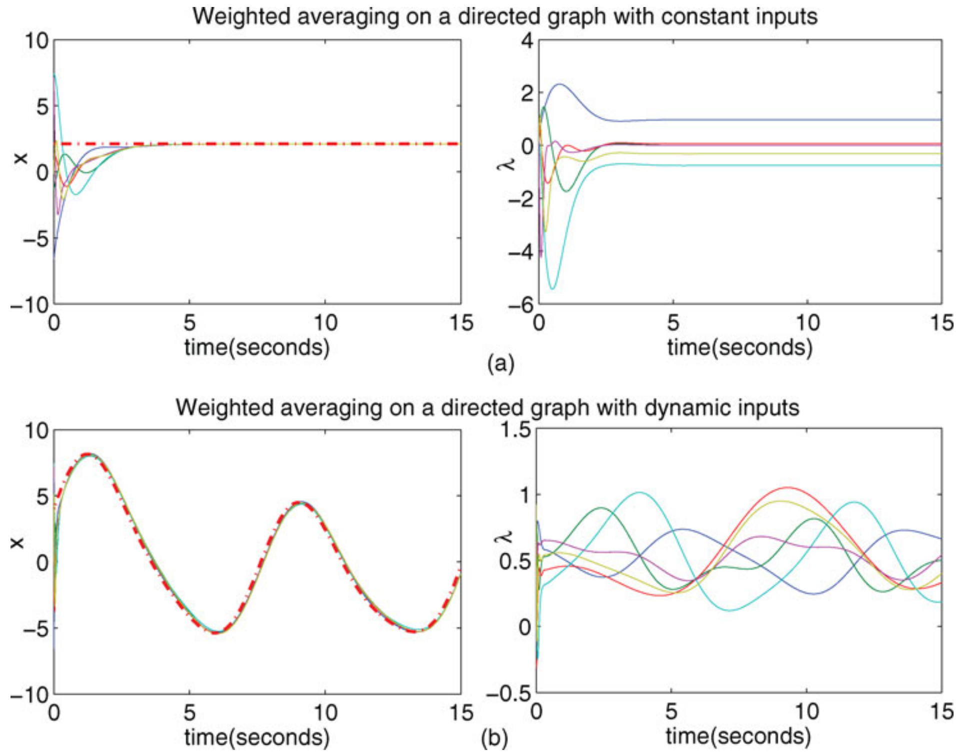

Figure 5. Weighted averaging on an imbalanced graph with (a) constant inputs and (b) dynamic inputs. The red dashed line represents the ideal weighted average.

7. Conclusions

In this paper, we solved two problems on distributed estimation: one is to estimate the direct average and the other is to estimate the weighted average of sensors' inputs over a sensor network modelled by a directed graph. We proposed new dynamic consensus filter design based on an existing PI consensus filter protocol, and removed the constraint on requiring bi-directional gain exchange between neighbouring sensors. Rigorous convergence analysis was conducted and sufficient gain conditions were obtained on the filter parameters to compensate for the asymmetry of corresponding Laplacian matrices. Simulations were performed in both cases of constant and time-varying inputs to show the effectiveness of the method. The proposed consensus filter design applies to real-world applications where sensor measurement accuracy or environmental conditions need to be taken into consideration in the final consensused group decision, such as the distributed cooperative spectrum sensing problem for radio networks that is described in the motivating example of the paper.

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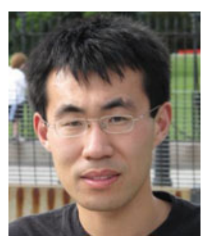
Notes on contributors



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Appendix A. Proof of Proposition 4.1

Consider A defined in (12). We have

$$A\begin{bmatrix}0_n\\ \mathbf{1}_n\end{bmatrix}=0_{2n},\ A^T\begin{bmatrix}0_n\\ \mathbf{1}_n\end{bmatrix}=0_{2n}, \tag{A1}$$

where 0_n, $\mathbf{1}_n$, and 0_{2n} denote a n-dimensional zero vector, a n-dimensional vector with all entries equal to 1, and a $2n$-dimensional zero vector, respectively. Without confusions, we omit the subscripts in (A1) hereafter. Equation (A1) indicates A has a zero eigenvalue at the vector $\begin{bmatrix}0\\ \mathbf{1}\end{bmatrix}$ for both the left eigenvector and the right eigenvector. With this property, we are able to partition the spectrum of A into two parts: the zero eigenvalue part and the non-zero eigenvalue part. To perform the partition, we define the similarity transformation matrix Q being an eigenvector matrix of the symmetric matrix:

$$\begin{bmatrix} I & (I-\frac{\mathbf{1}\mathbf{1}^T}{n})\\ (I-\frac{\mathbf{1}\mathbf{1}^T}{n}) & 0\end{bmatrix}. \tag{A2}$$

Note that this matrix has the same left zero eigenvector and the same right zero eigenvector as the matrix A, which is $[\,0^T\ \frac{\mathbf{1}^T}{\sqrt{n}}\,]$ and $\begin{bmatrix}0\\ \frac{\mathbf{1}}{\sqrt{n}}\end{bmatrix}$, respectively. According to the spectral theorem (Meyer 2001), there exists matrices $Q_1\in\mathbb{R}^{(2n-1)\times n}$ and $Q_2\in\mathbb{R}^{(2n-1)\times n}$ such that Q and Q^{-1} have the following forms,

$$Q=\begin{bmatrix}0 & \frac{\mathbf{1}^T}{\sqrt{n}}\\ Q_1 & Q_2\end{bmatrix},\ Q^{-1}=Q^T=\begin{bmatrix}0 & Q_1^T\\ \frac{\mathbf{1}}{\sqrt{n}} & Q_2^T\end{bmatrix}. \tag{A3}$$

Note that $Q^{-1}=Q^T$ holds since the matrix in (A2) is symmetric. Introduce the new variable z based on the following similarity transformation,

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Qe,$$

where $z \in \mathbb{R}^{2n}$, $z_1 \in \mathbb{R}$, and $z_2 \in \mathbb{R}^{2n-1}$. The dynamic of z can be obtained as follows:

$$\begin{aligned}\dot{z} &= Q\begin{bmatrix} -\gamma_1\Lambda L - \gamma_2 I & \gamma_3\Lambda L \\ -\Lambda L & 0 \end{bmatrix} Q^{-1}z - Q\begin{bmatrix} \dot{x}^* \\ \dot{\lambda}^* \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{\mathbf{1}^T}{n} \\ Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} -\gamma_1 L - \gamma_2 I & \gamma_3 L \\ -L & 0 \end{bmatrix}\begin{bmatrix} 0 & Q_1^T \\ \mathbf{1} & Q_2^T \end{bmatrix} z - Q\begin{bmatrix} \dot{x}^* \\ \dot{\lambda}^* \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} z - \begin{bmatrix} \frac{\beta^T\dot{\lambda}^*}{\mathbf{1}^T\mathbf{1}} \\ Q_1\dot{x}^* + Q_2\dot{\lambda}^* \end{bmatrix},\end{aligned} \tag{A4}$$

where $M \in \mathbb{R}^{(2n-1)\times(2n-1)}$ and $M = -\gamma_1 Q_1 L Q_1^T - \gamma_2 Q_1 Q_1^T - Q_2 L Q_1^T + \gamma_3 Q_1 L Q_2^T$. Separately, we have

$$\dot{z}_1 = -\frac{\mathbf{1}^T\dot{\lambda}^*}{n},\ \dot{z}_2 = Mz_2 - Q_1\dot{x}^* - Q_2\dot{\lambda}^*.$$

As to the consensus error measurement $e_1 = x - x^*$, according to the expression of Q^{-1} given in (A3), we have

$$e_1 = Q_1^T z_2.$$

Note that in z-coordinates, the system is completely decoupled into a one-dimensional invariant flow of z_1 and the rest $(2n-1)$-dimensional autonomous one in z_2. The eigenvalues of M are identical to the non-zero eigenvalues of the matrix A due to the similarity transformation. This completes the proof.

Appendix B. Proof of Proposition 4.2

We prove the result by analysing the transformed system (15) given in Proposition 4.1. For the case with the time-varying input $u(t)$, it is sufficient to prove that the system (15) with $\dot{x}^*$ and $\dot{\lambda}^*$ as inputs and e_1 as output is bounded-input bounded-output. According to the linear system theory (Chen 1998), this is equivalent to the statement that the matrix M has all eigenvalues strictly on LHP and the bound of the tracking error can be expressed as

$$\begin{aligned}\|e_1(t)\| \le{}& \frac{2\mu_{\max}^2(P)\|Q_1^T\|\|[Q_1,\, Q_2]\|c_0}{\mu_{\min}(P)} \\ &+ \|Q_1^T\|\|e_1(0)\|\sqrt{\frac{\mu_{\max}(P)}{\mu_{\min}(P)}},\end{aligned} \tag{B1}$$

where $t \geq 0$, c_0 represents the upper bound of the input, i.e., $\|[\dot{x}^{*T}\ \dot{\lambda}^{*T}]^T\| \le c_0$, P is the solution of the Lyapunov equation $PM + M^TP = -I$, $\mu_{\max}(P)$ and $\mu_{\min}(P)$ denote the largest and the smallest eigenvalues of P, respectively. Note that the second term on the right side of (B1) accounts for the zero input response, which attenuates to zero with time.

For the case with time-invariant input $u(t) = u(0)$, it is also sufficient to prove that the matrix M has all eigenvalues strictly on LHP as both x^* and λ^* are zero in this situation as pointed out in Remark 4.2.

In summary, for both the time-varying and the time-invariant cases, the statement that the matrix M has all eigenvalues strictly on LHP guarantees the conclusion. In addition, recalling the statement in Proposition 4.1 that the eigenvalues of M are identical to the non-zero eigenvalues of A defined in (12), we conclude the result if this matrix has all non-zero eigenvalues strictly on LHP. This completes the proof.

Appendix C. Proof of Theorem 5.1

The proof follows a similar procedure as the proof of Theorem 4.3 with some differences on the selection of the transformation matrix. We describe the procedure briefly as follows.

Step 1: Following a similar procedure in Section 4.2.1, the system can be equivalently transformed into the following:

$$\dot{e} = Ae - \begin{bmatrix} \dot{x}^* \\ \dot{\lambda}^* \end{bmatrix}, \tag{C1}$$

where

$$e = \begin{bmatrix} x \\ \lambda \end{bmatrix} - \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix},\ A = \begin{bmatrix} -\gamma_1\Lambda_1 L - \gamma_2 I & \gamma_3\Lambda_1 L \\ -\Lambda_1 L & 0 \end{bmatrix}, \tag{C2}$$

$$\begin{aligned}\dot{x}^* = \frac{\mathbf{1}\beta^T\dot{u}}{\mathbf{1}^T\beta},\ \dot{\lambda}^* &= \left(\begin{bmatrix} \gamma_3\Lambda_1 L \\ \mathbf{1}^T\Lambda_1^{-1} \end{bmatrix}^T \begin{bmatrix} \gamma_3\Lambda_1 L \\ \mathbf{1}^T\Lambda_1^{-1} \end{bmatrix}\right)^{-1} \\ &\times \begin{bmatrix} \gamma_3\Lambda_1 L \\ \mathbf{1}^T\Lambda_1^{-1} \end{bmatrix}^T \begin{bmatrix} \gamma_2(\frac{\mathbf{1}\beta^T}{\mathbf{1}^T\beta} - I)\dot{u} \\ \mathbf{1}^T\Lambda_1^{-1}\lambda_0 \end{bmatrix},\end{aligned} \tag{C3}$$

$$\begin{aligned}x^* = \frac{\mathbf{1}\beta^T u}{\mathbf{1}^T\beta},\ \lambda^* &= \left(\begin{bmatrix} \gamma_3\Lambda_1 L \\ \mathbf{1}^T\Lambda_1^{-1} \end{bmatrix}^T \begin{bmatrix} \gamma_3\Lambda_1 L \\ \mathbf{1}^T\Lambda_1^{-1} \end{bmatrix}\right)^{-1} \\ &\times \begin{bmatrix} \gamma_3\Lambda_1 L \\ \mathbf{1}^T\Lambda_1^{-1} \end{bmatrix}^T \begin{bmatrix} \gamma_2(\frac{\mathbf{1}\beta^T}{\mathbf{1}^T\beta} - I)u \\ \mathbf{1}^T\Lambda_1^{-1}\lambda_0 \end{bmatrix}.\end{aligned} \tag{C4}$$

Step 2: The eigenvalue, denoted as μ, of A satisfies

$$\det\left(\begin{bmatrix} -\gamma_1\Lambda_1 L - \gamma_2 I - \mu I & \gamma_3\Lambda_1 L \\ -\Lambda_1 L & -\mu I \end{bmatrix}\right) = 0. \tag{C5}$$

Noticing that the off-diagonal blocks commute, we equivalently get

$$\det(\gamma_3 L_0^2 + \mu^2 I + \mu\gamma_2 I + \mu\gamma_1 L_0) = 0, \tag{C6}$$

where $L_0 = \Lambda_1 L$. Since Λ_1 is a diagonal matrix with all positive diagonal elements, the off-diagonal elements of L_0 are not greater than 0 and its diagonal elements are all positive. Also, we have $L_0\mathbf{1} = \Lambda_1 L\mathbf{1} = 0$ and $\beta^T L_0 = \beta^T \operatorname{diag}^{-1}(\beta) L_0 = \mathbf{1}^T L = 0$, which mean L_0 is actually a Laplacian matrix with β being its left zero eigenvector. The characteristic Equation (C6) is a polynomial of L_0, so we get $(\gamma_3\eta^2 + \mu^2 I + \mu\gamma_2 + \mu\gamma_1\eta) = 0$ with η denoting the eigenvalue of L_0. Solving this equation yields

$$\mu_i = \frac{-(\gamma_2 + \gamma_1\eta_i) \pm \sqrt{(\gamma_2 + \gamma_1\eta_i)^2 - 4\gamma_3\eta_i^2}}{2}. \tag{C7}$$

Clearly, for $\eta_1 = 0$, $\mu_1 = 0$. Now, we only need to show that the rest eigenvalues of A locate at LHP. By following the same argument in the proof of Theorem 5.1, we can obtain that the statement in this theorem indeed guarantees $\mathrm{Re}(\mu_i) < 0$ for $i = 2, 3, \ldots, n$, which completes the proof.

Appendix D. Proof of Corollary 5.2

According to the proof in Theorem 5.1, we only need to prove that all non-zero eigenvalues of the $2n \times 2n$ matrix A defined in (C2) locate strictly on LHP. To prove the result, we equivalently show that the auxiliary linear autonomous system with A as the system matrix has $2n-1$ dimensional asymptotically stable flows, which correspond to the $2n-1$ eigenvalues on LHP and a single critically stable flow, which corresponds to the zero eigenvalue. The auxiliary linear system is constructed as follows:

$$\dot{y} = Ay, \tag{D1}$$

where A is defined as

$$A = \begin{bmatrix} -\gamma_1\Lambda_1 L - \gamma_2 I & \gamma_3\Lambda_1 L \\ -\Lambda_1 L & 0 \end{bmatrix}. \tag{D2}$$

We construct a Lyapunov function $V = y^T P y$ with $P = \mathrm{diag}(\frac{\Lambda_1^{-1}}{\gamma_3}, \Lambda_1^{-1})$. Then we have the following result by exploiting the property $L = L^T$ for an undirected graph

$$\dot{V} = -\, y^T \begin{bmatrix} \frac{2}{\gamma_3}(\gamma_1 L + \gamma_2\Lambda_1^{-1}) & 0 \\ 0 & 0 \end{bmatrix} y \leq 0. \tag{D3}$$

Clearly, $\gamma_1 L + \gamma_2\Lambda_1^{-1}$ is positive definite as $L = L^T$. We find the largest invariant set is $\mathbb{S}_1 = \{y = [y_1, y_2, \ldots, y_{2n}]^T \in \mathbb{R}^{2n}, y_1 = y_2 = \cdots = y_{2n-1} = 0, y_{2n} = k_0, \forall k_0 \in \mathbb{R}\}$ by letting $\dot{V} = 0$ in (D3). According to LaSalle's invariance principle, we can conclude that this auxiliary system converges to the invariant set $\mathbb{S}_1$ with a single degree of freedom $k_0 \in \mathbb{R}$, which in turn validates the result.