Single Particle Dynamics and Control in a Sliding Nanocluster System

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***Abstract*— We discuss single particle stability in a sliding nanocluster system represented by the Frenkel-Kontorova model. Both open-loop and closed-loop stability is studied using Lyapunov theory based methods. The systems are used to describe the frictional dynamics of a one-dimensional particle array sliding on a surface, which is subject to two potentials, namely, the substrate-particle and the inter-particle potentials. Average control can be applied to the system to control frictional properties in a desired way. We reveal that single particles are locally stable in the open-loop system without external forces. We also derive sufficient conditions so that the system under the average control law can be asymptotically stabilized. Simulation results are shown to verify the theoretical claims.**

I. Introduction

Many low-dimensional nonlinear physics can be modeled using the Frenkel-Kontorova (FK) model, which describes a chain of classical particles coupled to their neighbors and subject to a periodic on-site potential, see Figure 1 ([1]). The model characterizes the fundamental physics in problems such as sliding of nano-particle array, DNA dynamics, charge-density waves, magnetic spirals, and absorbed monolayers [4]. Particularly, it is one of the best known of simple models to characterize the microscopic mechanisms of friction ([5]). The study of friction is important from a practical point of view as it finds huge applications including those in micro-electro-mechanical systems (MEMS) and biological systems (such as the lubrication in joints). Recent advances have substantially improved the understanding of frictional phenomena, particularly on the inherently nonlinear nature of friction [14]. In controlling frictional properties in a desired way, it is traditionally achieved by chemical means, supplementing base lubricants by so-called friction modifier additives ([7], [16]). It is until recently that a different approach has attracted considerable interest, which is to control the system mechanically by applying small perturbations to accessible elements of the system [2], [6], [8].

While the FK model has been studied in physics for application purposes (see [4] and references therein), we have recently studied its dynamics and control using control theory based methods in [11], [12]. We presented tracking control algorithms so that the average velocity (the velocity of the center of mass) of the one-dimensional nanoarray tracks a constant targeted velocity. We used Lyapunov stability theory based method in the tracking control design, and open-loop stability of the interconnected particle system is mentioned without a rigorous proof. The control therein is for the average system, and single particles in the closed-loop system are not necessarily stable (and are actually unstable in many cases) though the average system is stable. In this paper, we focus on the single particle dynamic in both the open-loop and closed-loop systems, and derive sufficient conditions to achieve single particle stability under average controls. Rigorous proof will be provided using Lyapunov theory based methods.

The rest of the paper is organized as follows. Section II will present the FK model and its sliding control. In Section III, we prove the open-loop stability of local equilibrium points. Then, we discuss in Section IV the stability of single particles in the closed-loop system under average controls. Sufficient conditions will be given using Laypunov theory based methods. Simulation results are shown in Section V. Finally, we will conclude in Section VI.

II. The Frenkel-Kontorova Model and Sliding Control

The basic equations for the driven dynamics of a one dimensional particle array of N identical particles moving on a surface are given by a set of coupled nonlinear equations in [2], [11]. Upon the assumption of a sinusoidal substrate potential, the simplified equation of the Frenkel-Kontorova model is shown as:

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) \quad = \quad f + F_i \tag{1}$$

where ϕ_i is the dimensionless phase variable, $\phi_i = 2\pi x_i$, and F_i is the nearest-neighbor interaction force. A specific example often considered for F_i is the linearized Morse-type interaction ([2], [3]):

$$F_i \quad = \quad \kappa\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right) \tag{2}$$

where κ and β are positive constants. The free-end boundary conditions are:

$$F_1 = \kappa(\phi_2 - \phi_1), \qquad F_N = \kappa(\phi_{N-1} - \phi_N). \tag{3}$$

Control can be applied to the particle array , so that the frictional dynamics of a small array of particles is controlled towards preassigned values of the average sliding velocity. Let the external force, f, be a feedback control, denoted by $u(t)$. Rewrite the system model (1) as follows ([2]):

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) \quad = \quad F_i + u(t) \tag{4}$$

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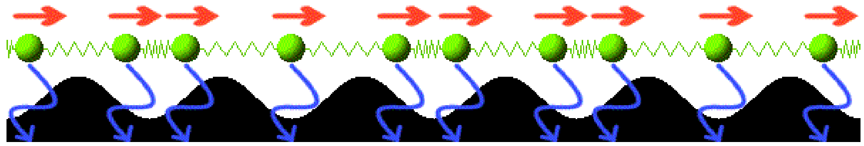

Fig. 1. The Frenkel-Kontorova model is a harmonic chain (mimic a layer of nano-particles) in a spatially periodic potential (mimic the substrate). The chain is driven by a constant force which is damped by a velocity-proportional damping.

Due to physical accessibility, the feedback control $u(t)$ is a function of three measurable quantities, $v_{target}, v_{c.m.}$, and $\phi_{c.m.}$, where v_{target} is the constant targeted velocity for the center of mass, $v_{c.m.}$ is the average (center of mass) velocity, *i.e.*,

$$v_{c.m.} = \frac{1}{N}\sum_{i=1}^{N}\dot{\phi}_i,$$

and $\phi_{c.m.}$ is the average (center of mass) position, *i.e.*,

$$\phi_{c.m.} = \frac{1}{N}\sum_{i=1}^{N}\phi_i.$$

The control objective for the sliding nanocluster system is to design a feasible feedback control law

$$u(t) = u(v_{target}, v_{c.m.}, \phi_{c.m.}), \tag{5}$$

such that $v_{c.m.}$ tends to v_{target}. The control law we proposed in [11] has the following form:

$$\begin{aligned} u(t) &= \gamma v_{target} + \sin v_{target}t - k_1(\phi_{c.m.} - v_{target}t) \\ &\quad -k_2(v_{c.m.} - v_{target}) \end{aligned} \tag{6}$$

where k_1, k_2 are positive constants. We proved in [11] that the average control law (6) renders the average closed-loop system bounded. It is easy to see from simulations that single particle dynamics are not necessarily stable. We conduct further study in this paper on the single particle stability under such an average control.

Remark 1: The feedback control law (6) takes the measurable average quantities $v_{c.m.}$ and $\phi_{c.m.}$ as inputs, which are the position and velocity of the center of mass. The model (4) represents the quartz-crystal microbalance (QCM) experiment for friction measurement where the applied force acts as an external variable that can be controlled experimentally [2]. Therefore, the proposed control law is implementable experimentally for friction control.

III. Local Stability of Single Particles

Before we discuss the single particle stability in the closed-loop system, we first examine the open-loop stability of single particles. The equilibrium points of the un-coupled particles without external force (*i.e.*, $f = 0$) are at

$$\phi_i = l\pi, \quad \dot{\phi}_i = 0, \quad l = 0, \pm 1, \pm 2, \dots \tag{7}$$

Express the dynamics in (1) without external forces in the following state space form:

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= -\sin x_{i1} - \gamma x_{i2} + F_i \end{aligned} \tag{8}$$

where $i = 1, 2 \dots, N$, $x_{i1} = \phi_i, x_{i2} = \dot{\phi}_i$, and F_i is the Morse-type particle interaction.

We consider the local stability of (8). Linearizing the system around its fixed points $(x_{i1}, x_{i2}) = (l\pi, 0)$, and stacking the state space equations for $i = 1, 2, \dots, N$, we obtain

$$\dot{x} = Ax + BFx \tag{9}$$

where $x = [x_{11}\ x_{12}\ x_{21}\ x_{22}\ \dots\ x_{N1}\ x_{N2}]^T$,

$$A = I_N \otimes A_i, \quad B = I_N \otimes B_i, \quad F = Q \otimes \begin{bmatrix} \kappa & 0 \end{bmatrix},$$

and

$$\begin{aligned} A_i &= \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}, \text{when } l = 2k\pi, \quad k = 0, \pm 1, \dots, \\ A_i &= \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix}, \text{when } l = (2k+1)\pi, \\ B_i &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ Q &= \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots \\ & & \vdots & & \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{bmatrix}. \end{aligned} \tag{10}$$

Since

$$\begin{aligned} BF &= (I_N \otimes B_i)(Q \otimes \begin{bmatrix} \kappa & 0 \end{bmatrix}) \\ &= (I_N Q) \otimes (B_i \begin{bmatrix} \kappa & 0 \end{bmatrix}) = Q \otimes \begin{bmatrix} 0 & 0 \\ \kappa & 0 \end{bmatrix}, \end{aligned}$$

the eigenvalues of BF are all zero.

We have the following theorem:

Theorem 1: In the absence of external forces, the system (1) with linear particle interaction (2) is asymptotically stable at the local equilibrium points $(2k\pi, 0)$, and it is unstable at $((2k+1)\pi, 0)$.

We need the following two Lemma to prove Theorem 1.

Lemma 1 ([9], page 171): If A is an $n \times n$ real symmetric matrix, then there are matrices L and D such that $L^TL = LL^T = I$ and $LAL^T = D$, where D is the diagonal matrix of eigenvalues of A.

Lemma 2 ([15], Appendix A): Define the set W consisting of all zero row sum matrices which have only nonpositive off-diagonal elements. A matrix $A \in W$ satisfies:

1) All eigenvalues of A are nonnegative;
2) 0 is an eigenvalue of A;
3) If A is irreducible, then 0 is an eigenvalue of multiplicity 1.

Proof of Theorem 1:

First, we study stability of the linearized system (9) for any positive constants γ, κ and for any $N \geq 2$. We find a transformation matrix to transform the system matrix into a block triangular one.

Since Q is a real symmetric matrix, according to Lemma 1, there exists a matrix T such that $T^{-1}QT = D$ where D is a diagonal matrix of eigenvalues of Q. Since $-Q \in W$ as defined in Lemma 2, Q have nonpositive eigenvalues (*i.e.*, one zero and all others negative). Therefore, D has nonpositive diagonal elements.

Let

$$\overline{T} \quad = \quad T \otimes I_2 \tag{11}$$

where I_2 is a 2×2 identity matrix. Then:

$$\begin{aligned} & \overline{T}^{-1}(A+BF)\overline{T} \\ = \quad & \left(T^{-1}I_N T\right) \otimes A_i + \left(T^{-1}QT\right) \otimes \left[\begin{array}{cc} 0 & 0 \\ \kappa & 0 \end{array}\right] \\ = \quad & I_N \otimes A_i + D \otimes \left[\begin{array}{cc} 0 & 0 \\ \kappa & 0 \end{array}\right] \\ \stackrel{def}{=} \quad & H \end{aligned} \tag{12}$$

We can see that H is block diagonal.

At the equilibrium points $(2k\pi, 0)$, the diagonal element is:

$$H_{ii} \quad = \quad \left[\begin{array}{cc} 0 & 1 \\ -1+\alpha_i\kappa & -\gamma \end{array}\right], \tag{13}$$

where $\alpha_i, i = 1, 2, \ldots, N$ are eigenvalues of Q. Since α_i are nonpositive, the eigenvalues of H_{ii} have negative real parts.

At the equilibrium points $((2k+1)\pi, 0)$, the diagonal element is:

$$H_{ii} \quad = \quad \left[\begin{array}{cc} 0 & 1 \\ 1+\alpha_i\kappa & -\gamma \end{array}\right]. \tag{14}$$

Its eigenvalues are

$$\frac{-\gamma \pm \sqrt{\gamma^2 + 4(1+\alpha_i\kappa)}}{2}.$$

Since 0 is a eigenvalue of matrix Q, *i.e.*, one of α_i is zero, H_{ii} has at least one positive eigenvalues.

Define a similarity transformation $x = Tz$. In the new coordinate, the system dynamics are

$$\dot{z} \quad = \quad Hz. \tag{15}$$

At the equilibrium points $(2k\pi, 0)$, the system (15) is asymptotically stable since H is block diagonal and the diagonal elements have negative eigenvalues. At the equilibrium points $((2k+1)\pi, 0)$, it is unstable. Therefore, the same stability result for the original system $\dot{x} = (A + BF)x$ can be obtained due to the similarity transformation. Furthermore, local stability of the original nonlinear system (1) can be obtained by the stability of its linearized system (9) ([13], Theorem 3.1). We conclude that the system (1) is asymptotically stable at $(2k\pi, 0)$, and unstable at other equilibrium points.

IV. Single Particle Stability in the Sliding Control System

To consider single particle stability in the closed-loop system, we define the following tracking error states:

$$e_{i1} = \phi_i - v_{target}t, \qquad e_{i2} = \dot{\phi}_i - v_{target}. \tag{16}$$

The corresponding error dynamics for single particles are given as:

$$\begin{aligned} \dot{e}_{i1} \quad &= \quad e_{i2} \\ \dot{e}_{i2} \quad &= \quad -\sin(e_{i1} + v_{target}t) - \gamma(e_{i2} + v_{target}) \\ & \qquad + F_i + u(t) \end{aligned} \tag{17}$$

Representing F_i using the error states, we have:

$$F_i \quad = \quad \kappa\left(e_{i+1,1} - 2e_{i1} + e_{i-1,1}\right). \tag{18}$$

Under the average control (6), we write the state space model of the closed-loop system in the following form:

$$\begin{aligned} \dot{e}_{i1} \quad &= \quad e_{i2} \\ \dot{e}_{i2} \quad &= \quad -\gamma e_{i2} + F_i - \bar{k}_1\left(\sum_{i=1}^{N} e_{i1}\right) - \bar{k}_2\left(\sum_{i=1}^{N} e_{i2}\right) \\ & \qquad + \left[\sin(v_{target}t) - \sin(e_{i1} + v_{target}t)\right] \end{aligned} \tag{19}$$

where $\bar{k}_1 = \frac{k_1}{N}, \bar{k}_2 = \frac{k_2}{N}$.

We re-present the system model in the following form:

$$\dot{E} \quad = \quad GE + f(e,t) \tag{20}$$

where $E = [e_{11}\ e_{12}\ e_{21}\ e_{22}\ \ldots\ e_{N1}\ e_{N2}]^T$,

$$\begin{aligned} G \quad &= \quad I_N \otimes \left[\begin{array}{cc} 0 & 1 \\ 0 & -\gamma \end{array}\right] \\ & \qquad + \left(I_N \otimes \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right)\left(Q \otimes \left[\begin{array}{cc} \kappa & 0 \end{array}\right]\right) \\ & \qquad + \left(I_N \otimes \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right)\left(\Theta \otimes \left[\begin{array}{cc} -\bar{k}_1 & -\bar{k}_2 \end{array}\right]\right) \\ &= \quad I_N \otimes \left[\begin{array}{cc} 0 & 1 \\ 0 & -\gamma \end{array}\right] + Q \otimes \left[\begin{array}{cc} 0 & 0 \\ \kappa & 0 \end{array}\right] \\ & \qquad + \Theta \otimes \left[\begin{array}{cc} 0 & 0 \\ -\bar{k}_1 & -\bar{k}_2 \end{array}\right], \end{aligned}$$

$$f(e,t) = \left[\begin{array}{c} 0 \\ \sin(v_{target}t) - \sin(e_{11} + v_{target}t) \\ 0 \\ \sin(v_{target}t) - \sin(e_{21} + v_{target}t) \\ \vdots \\ 0 \\ \sin(v_{target}t) - \sin(e_{N1} + v_{target}t) \end{array} \right],$$

Q is defined in (10), and Θ is the N by N matrix of ones.

We have the following lemma on the linear part of the system (20).

Lemma 3: There exists a similarity transformation such that the matrix G in (20) can be transformed to a block diagonal one.

Proof: Notice that the matrix $(-Q)$ is a real symmetric matrix with row sum zero, and it is irreducible. From Lemma 1 and 2, $(-Q)$ has eigenvalues

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{N-1} > \mu_N = 0.$$

It is always possible to choose the eigenvectors to be real, normalized and mutually orthogonal. Denote the eigenvectors corresponding to each of the eigenvalues:

$$\begin{aligned} v_k &= [v_{1k}\ v_{2k}\ \ldots\ v_{Nk}], k = 1, 2, \ldots, N-1, \\ v_N&. \end{aligned} \tag{21}$$

Then $V = [v_1\ v_2\ \ldots\ v_N]$ is an orthogonal matrix, *i.e.*, $VV^T = V^TV = I$, implying $V^T = V^{-1}$, and

$$\sum_{k=1}^{n} v_{ki}v_{kj} = \sum_{k=1}^{n} v_{ik}v_{jk} = \delta_{ij}, \tag{22}$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Because of $V^TQV = diag(\mu_1, \mu_2, \ldots, \mu_N)$, we further have

$$(-Q)_{ij} = \sum_{k=1}^{n} \mu_k v_{ik} v_{jk}. \tag{23}$$

Because the eigenvectors $v_k, k = 1, 2, \ldots, N-1$, are orthogonal to v_N, the following property holds:

$$\begin{aligned} \sum_{j=1}^{N} v_{jk} &= 0, k = 1, 2, \ldots, N-1, \\ v_N &= \frac{1}{\sqrt{N}}[1\ 1\ \ldots\ 1]^T. \end{aligned} \tag{24}$$

Therefore, we have:

$$V^{-1}QV = -D_Q \tag{25}$$

where D_Q is a diagonal matrix with the diagonal entry $\mu_i, i = 1, 2, \ldots, N$.

Due to property (24), the matrix V transforms the all 1's matrix Θ to a diagonal one as well:

$$\begin{aligned} V^{-1}\Theta V &= \left[(V^{-1}\Theta V)_{ik}\right] \\ &= \left[\left(\sum_{j=1}^{N} v_{ji} \right) \left(\sum_{j=1}^{N} v_{jk} \right) \right] = D_\Theta \end{aligned} \tag{26}$$

where D_Θ is a diagonal matrix with diagonal entry $(D_\Theta)_{ii} = 0, i = 1, 2, \ldots, N-1$, and $(D_\Theta)_{NN} = N$.

Choose the transformation matrix as $T = V \otimes I_2$. We have:

$$\begin{aligned} T^{-1}GT &= I_N \otimes \left[\begin{array}{cc} 0 & 1 \\ 0 & -\gamma \end{array} \right] - D_Q \otimes \left[\begin{array}{cc} 0 & 0 \\ \kappa & 0 \end{array} \right] \\ &\quad + D_\Theta \otimes \left[\begin{array}{cc} 0 & 0 \\ -\bar{k}_1 & -\bar{k}_2 \end{array} \right] \\ &= I_N \otimes C_i, \end{aligned} \tag{27}$$

where

$$C_i = \left\{ \begin{array}{l} \left[\begin{array}{cc} 0 & 1 \\ -\mu_i\kappa & -\gamma \end{array} \right], i = 1, 2, \ldots, N-1, \\ \left[\begin{array}{cc} 0 & 1 \\ -k_1 & -k_2 - \gamma \end{array} \right], i = N. \end{array} \right. \tag{28}$$

This completes the proof of the lemma.

From Lemma 3 and its proof, we see that the linear part of the system (20) is stable since its similarity transformation shows a stable system. This property is important in showing the stability of the nonlinear system (20).

We are now in the position to state the main theorem of this section.

Theorem 2: For system parameters γ and κ that satisfy

$$\begin{aligned} \kappa &> \frac{1}{\min_{i \leq N-1}(\mu_i)}, \\ \gamma &> \frac{1}{\sqrt{\min_{i \leq N-1}(\mu_i)\kappa - 1}} \end{aligned} \tag{29}$$

where $\mu_i, i = 1, \ldots, N-1$ are the positive eigenvalues of the matrix $(-Q)$, the average control (6) asymptotically stabilize the error system (20) if k_1 and k_2 are chosen to satisfy

$$k_1 \geq \kappa \min_{i \leq N-1}(\mu_i), \qquad k_2 \geq 0. \tag{30}$$

Proof: We use the classic Lyapunov theory to prove the stability of the error system (20). Define the matrix

$$P_i = \left[\begin{array}{cc} \frac{1}{2}(\varepsilon_i + \lambda_i^2) & \frac{1}{2}\lambda_i \\ \frac{1}{2}\lambda_i & \frac{1}{2} \end{array} \right], \tag{31}$$

where ε_i, λ_i are positive design parameters.

Then,

$$\begin{aligned} &C_i^T P_i + P_i C_i \\ = &\left\{ \begin{array}{l} \left[\begin{array}{cc} -\lambda_i\mu_i\kappa & \Delta_i \\ \Delta_i & -(\gamma - \lambda_i) \end{array} \right], i = 1, 2, \ldots, N-1, \\ \left[\begin{array}{cc} -\lambda_N k_1 & \Delta_N \\ \Delta_N & -(k_2 + \gamma - \lambda_N) \end{array} \right], \end{array} \right. \end{aligned} \tag{32}$$

where

$$\begin{aligned} \Delta_i &= \frac{1}{2}(\varepsilon_i + \lambda_i^2 - \lambda_i\gamma - \mu_i\kappa), i = 1, 2, \ldots, N-1, \\ \Delta_N &= \frac{1}{2}[\varepsilon_i + \lambda_i^2 - \lambda_i(k_2 + \gamma) - k_1]. \end{aligned} \tag{33}$$

To make $\Delta_i, i = 1, \ldots, N$ zero, we choose

$$\begin{aligned}
\lambda_i &< \gamma, && (34)\\
\varepsilon_i &= \lambda_i(\gamma - \lambda_i) + \mu_i \kappa > 0; \;\; i = 1, 2, \ldots, N-1, &&\\
\lambda_N &< \gamma + k_2, &&\\
\varepsilon_N &= \lambda_N(k_2 + \gamma - \lambda_N) + k_1 > 0. && (35)
\end{aligned}$$

Define the following Lyapunov function candidate:

$$W(t, e) = E^T H E, \quad (36)$$

where

$$\begin{aligned}
H &= TPT^{-1}\\
&= (V \otimes I_2)(I_N \otimes P_i)(V^{-1} \otimes I_2) = I_N \otimes P_i
\end{aligned}$$

We can see that $W(t, e)$ can be re-written as:

$$W(t, e) = \sum_{i=1}^{N} \left\{ \frac{\varepsilon}{2} e_{i1}^2 + \frac{1}{2}(\lambda_i e_{i1} + e_{i2})^2 \right\}. \quad (37)$$

Take the time derivative of $W(t, e)$ along the closed-loop dynamics (20). Denote

$$\dot{W}(t, e) = W_1 + W_2 \quad (38)$$

where W_1 is generated by the linear part of the dynamics, and W_2 is by the nonlinear part of the dynamics.

We have:

$$\begin{aligned}
W_1 &= E^T \left(G^T TPT^{-1} + TPT^{-1} G \right) E\\
&= E^T \left[T \left(T^{-1} G T \right)^T P T^{-1} + TP \left(T^{-1} G T \right) T^{-1} \right] E
\end{aligned}$$

Because of (27), we have

$$\begin{aligned}
W_1 &= E^T \left[T \left(I_N \otimes C_i \right)^T P T^{-1} + TP \left(I_N \otimes C_i \right) T^{-1} \right] E\\
&= E^T \left[I_N \otimes \left(C_i^T P_i + P_i C_i \right) \right] E \quad (39)
\end{aligned}$$

We see that $\left[I_N \otimes \left(C_i^T P_i + P_i C_i \right) \right]$ is a block diagonal matrix with diagonal entries as shown in (32).

Now considering W_2, from (37), we obtain:

$$\begin{aligned}
W_2 &= \sum_{i=1}^{N} (\lambda_i e_{i1} + e_{i2})[\sin(v_{target} t) - \sin(e_{i1} + v_{target} t)]\\
&\leq \sum_{i=1}^{N} \left\{ \left(\lambda_i + \frac{\tau_i}{2} \right) e_{i1}^2 + \frac{1}{2\tau_i} e_{i2}^2 \right\}\\
&= E^T \left(I_N \otimes \begin{bmatrix} \lambda_i + \frac{\tau_i}{2} & 0 \\ 0 & \frac{1}{2\tau_i} \end{bmatrix} \right) E \quad (40)
\end{aligned}$$

where we bounded $\|\sin(a)\|$ by $\|a\|$, and applied the inequality $2ab \leq \tau a^2 + \frac{1}{\tau} b^2$ (for $a, b \in \Re$), τ_i is a positive constant.

Substitute (39) and (40) into (38), we obtain:

$$\dot{W}(t, e) = E^T (I_N \otimes S_i) E \quad (41)$$

where

$$S_i = \begin{bmatrix} -\lambda_i \mu_i \kappa + \lambda_i + \frac{\tau_i}{2} & 0 \\ 0 & -(\gamma - \lambda_i) + \frac{1}{2\tau_i} \end{bmatrix}, i \leq N-1;$$

$$S_N = \begin{bmatrix} -\lambda_i k_1 + \lambda_i + \frac{\tau_i}{2} & 0 \\ 0 & -(k_2 + \gamma - \lambda_i) + \frac{1}{2\tau_i} \end{bmatrix}.$$

To make S_i negative definite, we check S_i for $i \leq N-1$ first. We need:

$$\begin{aligned}
-\lambda_i \mu_i \kappa + \lambda_i + \frac{\tau_i}{2} &< 0, && (42)\\
-(\gamma - \lambda_i) + \frac{1}{2\tau_i} &< 0 && (43)
\end{aligned}$$

Let $\mu_i \kappa > 1$, from (42), we get

$$\tau_i < 2[\lambda_i(\mu_i \kappa - 1)]. \quad (44)$$

From (43), we get

$$\tau_i > \frac{1}{2(\gamma - \lambda_i)}. \quad (45)$$

To make sure that τ_i exist, we need

$$\frac{1}{2(\gamma - \lambda_i)} < 2[\lambda_i(\mu_i \kappa - 1)] \quad (46)$$

which turns into a second-order inequality:

$$4(\mu_i \kappa - 1)\lambda_i^2 - 4\gamma(\mu_i \kappa - 1)\lambda_i + 1 < 0 \quad (47)$$

Since $0 < \lambda_i < \gamma$, we need

$$[4\gamma(\mu_i \kappa - 1)]^2 > 4 \cdot 4(\mu_i \kappa - 1), \quad (48)$$

which turns to

$$\gamma^2 > \frac{1}{\mu_i \kappa - 1}. \quad (49)$$

Therefore, the sufficient conditions for $S_i (i \leq N-1)$ to be negative definite are:

$$\begin{aligned}
\kappa &> \frac{1}{\min_{i \leq N-1}(\mu_i)},\\
\gamma &> \frac{1}{\sqrt{\min_{i \leq N-1}(\mu_i)\kappa - 1}} \quad (50)
\end{aligned}$$

Choosing the control parameters:

$$\begin{aligned}
k_1 &\geq \kappa \min_{i \leq N-1}(\mu_i)\\
k_2 &\geq 0, \quad (51)
\end{aligned}$$

S_N will be negative definite.

The proof of the theorem follows directly from Lyapunov theory because of the positive definiteness of W in (36) and the negative definiteness of $\dot{W}$ in (41).

V. Simulation Results

We have performed numerical simulations on arrays of different sizes $(3 \leq N \leq 256)$. First, we verified that the set of fixed points of un-forced frictional dynamics, $(\phi_i, \dot{\phi}_i) = (l\pi, 0)$ with l even numbers, are locally stable. This can be seen from Fig. 2, where a 5-particle system is simulated. The system parameters are $\gamma = 0.1, \kappa = 0.26$ ([2], [10]). Random initial conditions are used in the simulations. Local stability of individual particles in the five-particle system can be observed in the figure.

We show the single particle dynamics in the closed-loop system under the average control (6) in Fig. 3. The system parameters used are $\gamma = 4, \kappa = 1.5$ for a three-particle interconnected system. The control parameters are chosen to be $k_1 = 5.8, k_2 = 4$. The targeted velocity is $v_{target} = 1.5$. Fig. 3 shows that the tracking error for each individual particle tends to zero which indicates that the velocity of each particle in the interconnected system tracks the targeted value. This verifies the result in Theorem 4.

VI. Conclusions

We studied the single particle stability for a one-dimensional particle array sliding on a surface subject to friction. The well-known Frenkel-Kontorova model is used to describe the dynamics, which is a nonlinear interconnected system. The feedback control uses the average quantity of the system which are physically accessible. It was revealed first that a set of equilibrium points of the un-forced system is asymptotically stable. Then, we derived sufficient conditions for the average control to stabilize single particles in the interconnected tracking error system. The Lyapunov theory based method is utilized in the stability analysis. Simulation results were shown to verify the theoretical claim. The results of the paper can be applied to other physical systems whose dynamics can be described by the Frenkel-Kontorova model.

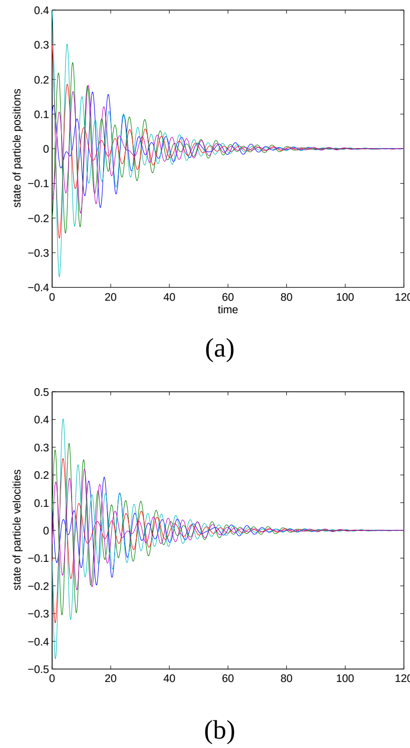

Fig. 2. Local stability of single particles in the open-loop system: (a) the phase variable, (b) the velocity variable.

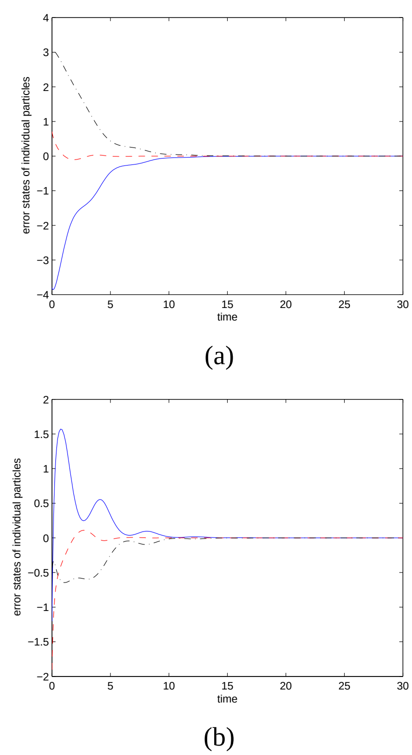

Fig. 3. Single particle dynamics in the closed-loop system for targeted value $v_{target} = 1.5$: (a) the phase variables, (b) the velocity variables.

References

[1] http://monet.unibas.ch/ elmer/flab/FKModel/index.html.

[2] Y. Braiman, J. Barhen, and V. Protopopescu. Control of friction at the nanoscale. *Physical Review Letters*, 90(9), 2003.

[3] Y. Braiman, F. Family, and H. G. E. Hentschel. Nonlinear friction in the periodic stick-slip motion of coupled oscillators. *Physical Review B*, 55(5):5491–5504, 1997.

[4] O. M. Braun and Y. S. Kivshar. *The Frenkel-Kontorova Model*. Springer, Berlin Heidelberg, 2004.

[5] O.M. Braun and A. G. Naumovets. Nanotribology: Microscopic mechanism of friction. *Surface Science Reports*, 60:79–158, 2006.

[6] A. Cochard, L. Bureau, and T. Baumberger. Stabilization of frictional sliding by normal load modulation: A bifurcation analysis. *Trans. ASME*, 70:220–226, 2003.

[7] O.K. Dudko, A.E. Filippov, J. Klafter, and M. Urbakh. Chemical control of friction: Mixed lubricant monolayers. *Tribology Letters*, 12(4):217–227, 2002.

[8] J.P. Gao, W.D. Luedtke, and U. Landman. Friction control in thin-film lubrication. *J. Phys. Chem. B*, 102:5033–5037, 1998.

[9] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.

[10] Y. Guo and Z. Qu. Stabilization and tracking control of friction dynamics of a one-dimensional nanoarray. In *Proceedings of the American Control Conference*, pages 2487–2492, Portland, Oregon, June 2005.

[11] Y. Guo and Z. Qu. Feedback control of frictional dynamics. In *Proceedings of IEEE Conference on Decision and Control*, San Diego, CA, Dec 2006.

[12] Y. Guo, Z. Qu, and Z. Zhang. Lyapunov stability and precise control of the frictional dynamics of a one-dimensional nanoarray. *Physical Review B*, 73(9):094118, 2006.

[13] J. J. E. Slotine and W. Li. *Applied Nonlinear Control*. Prentics Hall, Nre Jersey, 1991.

[14] M. Urbakh, J. Klafter, D. Gourdon, and J. Israelachvili. The nonlinear nature of friction. *Nature*, 430:525–528, 2004.

[15] C. W. Wu. *Synchronization in Coupled Chaotic Circuits and Systems*. World Scientific, Singapore, 2002.

[16] Y. Zhu, H. Ohtani, M. L. Greenfield, M. Ruths, and S. Granick. Modification of boundary lubrication by oil-soluble friction modifier additives. *Tribology Letters*, 15(2):127–134, 2003.